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Brackets, sigma models and integrability of generalized complex structures

Sebastian Guttenberg

Service de Physique Théorique, CEA/Saclay
91191 Gif-sur-Yvette Cedex, France, and
Institut für Theoretische Physik, Technische Universität Wien,
1040 Wien, Austria
E-mail: basti@hep.itp.tuwien.ac.at

Abstract: It is shown how derived brackets naturally arise in sigma-models via Poisson- or antibracket, generalizing a recent observation by Alekseev and Strobl. On the way to a precise formulation of this relation, an explicit coordinate expression for the derived bracket is obtained. The generalized Nijenhuis tensor of generalized complex geometry is shown to coincide up to a de-Rham closed term with the derived bracket of the structure with itself and a new coordinate expression for this tensor is presented. The insight is applied to two known two-dimensional sigma models in a background with generalized complex structure.

Keywords: Sigma Models, Differential and Algebraic Geometry, Topological Field Theories, Superstrings and Heterotic Strings.

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1. Introduction

There are quite a lot of different geometric brackets floating around in the literature, like Schouten bracket, Nijenhuis bracket or in generalized complex geometry the Dorfman bracket and Courant bracket, to list just some of them. They are often related to integrability conditions for some structures on manifolds. The vanishing of the Nijenhuis bracket of a complex structure with itself, for example, is equivalent to its integrability. The same is true for the Schouten bracket and a Poisson structure. The above brackets can be unified with the concept of derived brackets [1]. Within this concept, they are all just natural extensions of the Lie-bracket of vector fields to higher rank tensor fields.

It is well known that the antibracket appearing in the Lagrangian formalism for sigma models is closely related to the Schouten-bracket in target space. In addition it was recently observed by Alekseev and Strobl that the Dorfman bracket for sums of vectors and one-forms appears naturally in two dimensional sigma models,[1] [2]. This was generalized by Bonelli and Zabzine [4] to a derived bracket for sums of vectors and p-forms on a p-brane.[2] These observations lead to the natural question whether there is a general relation between the sigma-model Poisson bracket or antibracket and derived brackets in target space. Working out the precise relation for sigma models with a special field content but undetermined dimension and dynamics, is the major subject of the present paper.

One of the motivations for this article, was the application to generalized complex geometry. The importance of the latter in string theory is due to the observation that effective spacetime supersymmetry after compactification requires the compactification manifold to be a generalized Calabi-Yau manifold [5, 3, 6 – 9]. Deviations from an ordinary Calabi Yau manifold are due to fluxes and also the concept of mirror symmetry can be generalized in this context. There are numerous other important articles on the subject, like e.g. [10 – 14] and many more. A more complete list of references can be found in [9]. A major part of the considerations so far was done from the supergravity point of view. Target space supersymmetry is, however, related to an $N = 2$ supersymmetry on the worldsheet. For this reason the relation between an extended worldsheet supersymmetry and the presence of an integrable generalized complex structure (GCS) was studied in [15] (the reviews [16, 17] on generalized complex geometry have this relation in mind). Zabzine clarified in [18] the relation in a model independent way in a Hamiltonian description and showed that the existence of a second non-manifest worldsheet supersymmetry $\boldsymbol{Q}_2$ in an $N = 1$ sigma-model is equivalent to the existence of an integrable GCS $\mathcal{J}$. It is the observation that the integrability of the GCS $\mathcal{J}$ can be written as the vanishing of a generalized bracket $[\mathcal{J},\mathcal{J}]_B = 0$ which leads to the natural question, whether there is a direct mapping between $[\mathcal{J},\mathcal{J}]_B = 0 \,\&\, \mathcal{J}^2 = -1$ on the one side and $\{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} = 2P$ on the other side. This will be a natural application in subsection 3.2 of the more general preceding considerations about

[1] In [2], the non-symmetric bracket is called 'Courant bracket'. Following e.g. Gualtieri [3] or [1] it will be called 'Dorfman bracket' in this paper, while 'Courant bracket' is reserved for its antisymmetrization (see (C.31) and (C.38)).

[2] The *Vinogradov bracket* appearing in [4] is just the antisymmetrization of a derived bracket (see footnote 25 on page 49).

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the relation between (super-)Poisson brackets in sigma models with special field content and derived brackets in the target space.

A second interesting application is Zucchini's Hitchin-sigma-model [19]. There are two more papers on that subject [20, 21], but the present discussion refers only to the first one. Zucchini's model is a two dimensional sigma-model in a target space with a generalized complex structure (GCS). The sigma-model is topological when the GCS is integrable, while the inverse does not hold. The condition for the sigma model to be topological is the master equation $(S,S) = 0$. Again we might wonder whether there is a direct mapping between the antibracket and S on the one hand and the geometric bracket and $\mathcal{J}$ on the other hand and it will be shown in subsection 3.1 how this mapping works as an application of the considerations in subsection 2.5. In order to understand more about geometric brackets in general, however, it was necessary to dive into Kosmann-Schwarzbach's review on derived brackets [1] which led to observations that go beyond the application to the integrability of a GCS .

The structure of the paper is as follows: The general relation between sigma models and derived brackets in target space will be studied in the next section. The necessary geometric setup will be established in 2.1. Although there are no new deep insights in 2.1, the unconventional idea to extend the exterior derivative on forms to multivector valued forms (see (2.32) and (2.35)) will provide a tool to write down a coordinate expression for the general derived bracket between multivector valued forms (2.48) which to my knowledge does not yet exist in literature. The main results in section 2, however, are the propositions 1 on page 18 and 1b on page 34 for the relation between the Poisson-bracket in a sigma-model with special field content and the derived bracket in the target space, and the proposition 3b on page 25 for the relation between the antibracket in a sigma-model and the derived bracket in target space. Proposition 2 on page 21 is just a short quantum consideration which only works for the particle case. In section 3 the propositions 1b and 3b are finally applied to the two examples which were mentioned above.

Another result is the relation between the generalized Nijenhuis tensor and the derived bracket of $\mathcal{J}$ with itself, given in (3.12). The derivation of this can be found in the appendix on page 61. In addition to this, there is a new coordinate form of the generalized Nijenhuis tensor presented in (C.58) on page 59, which might be easier to memorize than the known ones. There is also a short comment in footnote 28 on page 55 on a possible relation to Hull's doubled geometry.

Appendix A summarizes the used conventions, while appendix B is an introduction to geometric brackets. Finally, appendix C provides some aspects of generalized complex geometry which might be necessary to understand the two applications of above.

2. Sigma-model-induced brackets

2.1 Geometric brackets in phase space formulation

In the following some basic geometric ingredients which are necessary to formulate derived brackets will be given. Although there is no sigma model and no physics explicitly involved in this first subsection, the presentation and the techniques will be very suggestive, s.th.

there is visually no big change when we proceed after that with considerations on sigma-models.

2.1.1 Algebraic brackets

Consider a real differentiable manifold M. The interior product with a vector field $v = v^k \boldsymbol{\partial}_k$ (in a local coordinate basis) acting on a differential form ρ is a differential operator in the sense that it differentiates with respect to the basis elements of the cotangent space:[3]

$$\imath_v \rho^{(r)} = r \cdot v^k \rho^{(r)}_{k m_1 \ldots m_{r-1}}(x)\, \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_{r-1}} = v^k \frac{\partial}{\partial(\mathbf{d}x^k)} \left(\rho_{m_1 \ldots m_r} \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_r}\right) \quad (2.1)$$

Let us rename[4]

$$\boldsymbol{c}^m \equiv \mathbf{d}x^m \quad (2.2)$$

$$\boldsymbol{b}_m \equiv \boldsymbol{\partial}_m \quad (2.3)$$

The vector v takes locally the form $v = v^m \boldsymbol{b}_m$ and when we introduce a canonical graded Poisson bracket between $\boldsymbol{c}^m$ and $\boldsymbol{b}_m$ via $\{\boldsymbol{b}_m, \boldsymbol{c}^n\} = \delta_m^n$, we get

$$\imath_v \rho = \{v, \rho\} \quad (2.4)$$

Extending also the local x-coordinate-space to a phase space by introducing the conjugate momentum p_m (whose geometric interpretation we will discover soon), we have altogether the (graded) Poisson bracket

$$\{\boldsymbol{b}_m, \boldsymbol{c}^n\} = \delta_m^n = \{\boldsymbol{c}^n, \boldsymbol{b}_m\} \quad (2.5)$$

$$\{p_m, x^n\} = \delta_m^n = -\{x^n, p_m\} \quad (2.6)$$

$$\{A, B\} = A \frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_k} \frac{\partial}{\partial \boldsymbol{c}^k} B + A \frac{\overleftarrow{\partial}}{\partial p_k} \frac{\partial}{\partial x^k} B - (-)^{AB} \left(B \frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_k} \frac{\partial}{\partial \boldsymbol{c}^k} A + B \frac{\overleftarrow{\partial}}{\partial p_k} \frac{\partial}{\partial x^k} A \right) \quad (2.7)$$

and can write the exterior derivative acting on forms as generated via the Poisson-bracket by an odd phase-space function $\boldsymbol{o}(\boldsymbol{c}, p)$

$$\boldsymbol{o} \equiv \boldsymbol{o}(\boldsymbol{c}, p) \equiv \boldsymbol{c}^k p_k \quad (2.8)$$

$$\left\{\boldsymbol{o}, \rho^{(r)}\right\} = \boldsymbol{c}^k \{p_k, \rho_{m_1 \ldots m_r}(x)\} \boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_r} = \mathbf{d}\rho^{(r)} \quad (2.9)$$

The variables $\boldsymbol{c}^m$,$\boldsymbol{b}_m$,x^m and p_m can be seen as coordinates of $T^*(\Pi TM)$, the cotangent bundle of the tangent bundle with parity inversed fiber.

[3]Note, that a convention is used, were the prefactor $\frac{1}{r!}$ which usually comes along with an r-form is absorbed into the definition of the wedge-product. The common conventions can for all equations easily be recovered by redefining all coefficients appropriately, e.g. $\rho_{m_1 \ldots m_r} \to \frac{1}{r!} \rho_{m_1 \ldots m_r}$.

[4]The similarity with ghosts is of course no accident. It is well known (see e.g. [22]) that ghosts in a gauge theory can be seen as 1-forms dual to the gauge-vector fields and the BRST differential as the sum of the Koszul-Tate differential (whose homology implements the restriction to the constraint surface) and the longitudinal exterior derivative along the constraint surface. In that sense the present description corresponds to a topological theory, where all degrees of freedom are gauged away. But we will not necessarily always view $\boldsymbol{c}^m$ as ghosts in the following. So let us in the beginning see $\boldsymbol{c}^m$ just as another name for $\mathbf{d}x^m$. We do not yet assume an underlying sigma-model, i.e. $\boldsymbol{b}_m$ and $\boldsymbol{c}^m$ do not necessarily depend on a worldsheet variable.

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Interior product and "quantization". Given a multivector valued form $K^{(k,k')}$ of form degree k and multivector degree k', it reads in the local coordinate patch with the new symbols

$$K^{(k,k')} \equiv K^{(k,k')}(x,\boldsymbol{c},\boldsymbol{b}) \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(x)\,\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\boldsymbol{b}_{n_1}\cdots\boldsymbol{b}_{n_{k'}} \equiv K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{2.10}$$

The notation $K(x,\boldsymbol{c},\boldsymbol{b})$ should stress, that K is locally a (smooth on a C^∞ manifold) function of the phase space variables which will later be used for analytic continuation (x will be allowed to take c-number values of a superfunction). The last expression in the above equation introduces a *schematic index notation* which is useful to write down the explicit coordinate form for lengthy expressions. See in the appendix A at page 39 for a more detailed description of its definition. It should, however, be self-explanatory enough for a first reading of the article

One can define a natural generalization of the interior product with a vector $\imath_v$ to an *interior product* with a multivector valued form $\imath_K$ acting on some r-form (in fact, it is more like a combination of an interior and an exterior product — see footnote 23 on page 46 –, but we will stick to this name)

$$\imath_{K^{(k,k')}}\rho^{(r)} \equiv (k')!\begin{pmatrix} r \\ k' \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}}\rho_{\underbrace{l_{k'}\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}_{r}} = \tag{2.11}$$

$$= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\left\{\boldsymbol{b}_{n_1},\left\{\cdots,\left\{\boldsymbol{b}_{n_{k'}},\rho^{(r)}\right\}\right\}\right\} \tag{2.12}$$

$$= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\frac{\partial}{\partial\boldsymbol{c}^{n_1}}\cdots\frac{\partial}{\partial\boldsymbol{c}^{n_{k'}}}\rho^{(r)} \tag{2.13}$$

It is a derivative of order k' and thus not a derivative in the usual sense like $\imath_v$. The third line shows the reason for the normalization of the first line, while the second line is added for later convenience. The interior product is commonly used as an *embedding* of the multivector valued forms in the space of differential operators acting on forms, i.e. $K \to \imath_K$, s.th. structures of the latter can be induced on the space of multivector valued forms. In (2.13) the interior product $\imath_K$ can be seen, up to a factor of $\hbar/i$, as the quantum operator corresponding to K, where the form ρ plays the role of a wave function. The natural ordering is here to put the conjugate momenta to the right. We can therefore fix the following "*quantization*" rule (corresponding to $\hat{\boldsymbol{b}} = \frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{c}}$)

$$\hat{K}^{(k,k')} \equiv \left(\frac{\hbar}{i}\right)^{k'}\imath_{K^{(k,k')}} \tag{2.14}$$

$$\text{with } \imath_{K^{(k,k')}} = K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{n_1\ldots n_{k'}}\frac{\partial^{k'}}{\partial\boldsymbol{c}^{n_1}\cdots\partial\boldsymbol{c}^{n_{k'}}} \tag{2.15}$$

The (graded) commutator of two interior products induces an algebraic bracket due to Buttin [23], which is defined via

$$\left[\imath_{K^{(k,k')}},\imath_{L^{(l,,l')}}\right] \equiv \imath_{[K,L]^{\Delta}} \tag{2.16}$$

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Using the obvious generalization of $\partial_x^n(f(x)g(x)) = \sum_{p=0}^n \binom{n}{p} \partial_x^p f(x)\partial_x^{n-p}g(x)$, a short calculation leads to

$$\imath_K \imath_L = \sum_{p\geq 0} \imath_{\imath_K^{(p)} L} = \imath_{K\wedge L} + \sum_{p\geq 1} \imath_{\imath_K^{(p)} L} \tag{2.17}$$

where we introduced the *interior product of order p*

$$\imath^{(p)}_{K^{(k,k')}} \equiv \binom{k'}{p} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} \frac{\partial^p}{\partial \boldsymbol{c}^{n_1}\cdots\partial\boldsymbol{c}^{n_p}} \tag{2.18}$$

$$= \frac{1}{p!} K \frac{\overleftarrow{\partial}^p}{\partial \boldsymbol{b}_{n_p}\cdots\partial\boldsymbol{b}_{n_1}} \frac{\partial^p}{\partial \boldsymbol{c}^{n_1}\cdots\partial\boldsymbol{c}^{n_p}} \tag{2.19}$$

$$\Rightarrow \imath^{(p)}_{K^{(k,k')}} L^{(l,l')} = (-)^{(k'-p)(l-p)} p! \binom{k'}{p}\binom{l}{p} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} L_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{2.20}$$

which contracts only p of all k' upper indices and therefore coincides with the interior product of above when acting on forms for $p = k'$ and with the wedge product for $p = 0$.

$$\imath^{(k')}_{K^{(k,k')}}\rho = \imath_{K^{(k,k')}}\rho, \qquad \imath^{(0)}_K L = K\wedge L \tag{2.21}$$

Using (2.17) the *algebraic bracket* $[\,,\,]^\Delta$ defined in (2.16) can thus be written as

$$[K^{(k,k')}, L^{(l,l')}]^\Delta = \sum_{p\geq 1} \underbrace{\imath_K^{(p)} L - (-)^{(k-k')(l-l')}\imath_L^{(p)} K}_{\equiv [K,L]^\Delta_{(p)}} \tag{2.22}$$

(2.20) provides the explicit coordinate form of this algebraic bracket. From (2.19) we recover the known fact that the $p = 1$ term of the algebraic bracket is induced by the Poisson-bracket and therefore is itself an algebraic bracket, called the *big bracket* [1] or *Buttin's algebraic bracket* [23]

$$[K,L]^\Delta_{(1)} = \imath_K^{(1)} L - (-)^{(k-k')(l-l')}\imath_L^{(1)} K$$

$$\stackrel{(2.19)}{=} \{K, L\} \tag{2.23}$$

$$\stackrel{(2.20)}{=} (-)^{(k'-1)(l-1)} k' l\, K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1} L_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{2.24}$$

$$-(-)^{(k-k')(l-l')}(-)^{(l'-1)(k-1)} l' k\, L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1} K_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}$$

For $k' = l' = 1$ it reduces to the Richardson-Nijenhuis bracket (B.60) for vector valued forms. In [1] the big bracket is described as the canonical Poisson structure on $\bigwedge^\bullet(T\oplus T^*)$ which matches with the observation in (2.23). The bracket takes an especially pleasant coordinate form for generalized multivectors as is presented in equation (C.67) on page 61.

The multivector-degree of the p-th term of the complete algebraic bracket (2.22) is $(k' + l' - p)$, so that we can rewrite (2.16) in terms of "quantum"-operators (2.14) in the following way:

$$\left[\hat{K}^{(k,k')}, \hat{L}^{(l,l')}\right] = \sum_{p\geq 1}\left(\frac{\hbar}{i}\right)^p \widehat{[K,L]^\Delta_{(p)}} \tag{2.25}$$

$$= \left(\frac{\hbar}{i}\right)\widehat{\{K,L\}} + \sum_{p\geq 2}\left(\frac{\hbar}{i}\right)^p \widehat{[K,L]^\Delta_{(p)}} \tag{2.26}$$

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The Poisson bracket is, as it should be, the leading order of the quantum bracket.

2.1.2 Extended exterior derivative and the derived bracket of the commutator

In the previous subsection the commutator of differential operators induced (via the interior product as embedding) an algebraic bracket on the embedded tensors. Also other structures from the operator space can be induced on the tensors. Having the commutator at hand, one can build the *derived bracket* (see footnote 20 on page 44) of the commutator by additionally commuting the first argument with the exterior derivative. Being interested in the induced structure on multivector valued forms, we consider as arguments only interior products with those multivector valued forms

$$[\imath_K,_{\mathbf{d}}\imath_L] \equiv [[\imath_K, \mathbf{d}], \imath_L] \tag{2.27}$$

One can likewise use other differentials to build a derived bracket, e.g. the twisted differential $[\mathbf{d} + H, \ldots]$ with an odd closed form H, which leads to so called twisted brackets. Let us restrict to $\mathbf{d}$ for the moment. The derived bracket is in general not skew-symmetric but it obeys a graded Jacobi-identity and is therefore what one calls a Loday bracket. When looking for new brackets, the Jacobi identity is the property which is hardest to check. A mechanism like above, which automatically provides it is therefore very powerful. The above derived bracket will induce brackets like the Schouten bracket or even the Dorfman bracket of generalized complex geometry on the tensors. In general, however, the interior products are not closed under its action, i.e. the result of the bracket cannot necessarily be written as $\imath_{\tilde{K}}$ for some $\tilde{K}$. An expression for a general bracket on the tensor level, which reduces in the corresponding cases to the well known brackets therefore does not exist. Instead one normally has to derive the brackets in the special cases separately. In the following, however, a natural approach is discussed including the new variable p_m, introduced in (2.6), which leads to an explicit coordinate expression for the general bracket. This expression is of course tensorial only in the mentioned special cases, that is when terms with p_m vanish. This is not an artificial procedure, as the conjugate variable p_m to x^m is always present in sigma-models, and it will in turn explain the geometric meaning of p_m.

The exterior derivative $\mathbf{d}$ acting on forms is usually not defined acting on multivector valued forms (otherwise we could build the derived bracket of the algebraic bracket (2.22) by $\mathbf{d}$ without lifting everything to operators via the interior product). But via $\{\boldsymbol{o}, K^{(k,k')}\}$ we can, at least formally, define a differential on multivector valued forms. The result, however, contains the variable p_k which we have not yet interpreted geometrically. After extending the definition of the interior product to objects containing p_m, we will get the relation $[\mathbf{d}, \imath_K] = \imath_{\{\boldsymbol{o},K\}}$, i.e. $\{\boldsymbol{o}, \ldots\}$ can be seen as an induced differential from the space of operators. For forms $\omega^{(q)}$, this simply reads $[\mathbf{d}, \imath_\omega] = \imath_{\mathbf{d}\omega}$. The definition $\mathbf{d}K \equiv \{\boldsymbol{o}, K\}$ thus seems to be a reasonable extension of the exterior derivative to multivector valued forms. Let us first provide the necessary definitions.

Consider a phase space function, which is of arbitrary order in the variable p_k

$$T^{(t,t',t'')}(x, \boldsymbol{c}, \boldsymbol{b}, p) \equiv T_{m_1 \ldots m_t}{}^{n_1 \ldots n_{t'} k_1 \ldots k_{t''}}(x)\, \boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_t} \boldsymbol{b}_{m_1} \cdots \boldsymbol{b}_{m_{t'}} p_{k_1} \cdots p_{k_{t''}} \tag{2.28}$$

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T is symmetrized in $k_1 \dots k_{t''}$,while it is antisymmetrized in the remaining indices. Using the usual quantization rules $\boldsymbol{b} \to \frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{c}}$ and $p \to \frac{\hbar}{i}\frac{\partial}{\partial x}$ with the indicated ordering (conjugate momenta to the right) while still insisting on (2.14) as the relation between quantum operator and interior product, we get an extended definition of the *interior product* (2.12), (2.13):

$$\imath_{T^{(t,t',t'')}} \equiv \left(\frac{i}{\hbar}\right)^{t'+t''} \hat{T}^{(t,t',t'')} \equiv T_{m_1 \dots m_t}{}^{n_1 \dots n_{t'} k_1 \dots k_{t''}} \boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_t} \frac{\partial^{t'}}{\partial \boldsymbol{c}^{n_1} \cdots \partial \boldsymbol{c}^{n_{t'}}} \frac{\partial^{t''}}{\partial x^{k_1} \cdots \partial x^{k_{t''}}} \tag{2.29}$$

$$\begin{aligned} ip_{T^{(t,t',t'')}} \rho^{(r)} &= T_{m_1 \dots m_t}{}^{n_1 \dots n_{t'} k_1 \dots k_{t''}} \boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_t} \times \\ &\qquad \times \Big\{ \boldsymbol{b}_{n_1}, \Big\{ \cdots, \big\{ \boldsymbol{b}_{n_{t'}}, \{ p_{k_1}, \{ \cdots, \{ p_{k_{t''}}, \rho^{(r)} \}\}\}\big\}\Big\}\Big\} &(2.30)\\ &= (t')! \binom{r}{t'} T_{\boldsymbol{m} \dots \boldsymbol{m}}{}^{n_1 \dots n_{t'} k_1 \dots k_{t''}} \frac{\partial^{t''}}{\partial x^{k_1} \cdots \partial x^{k_{t''}}} \rho^{(r)}_{n_{t'} \dots n_1 \boldsymbol{m} \dots \boldsymbol{m}} &(2.31) \end{aligned}$$

The operator $\imath_T$ will serve us as an embedding of T (a phase space function, which lies in the extension of the space of multivector valued forms by the basis element p_k) into the space of differential operators acting on forms. Because of the partial derivatives with respect to x, the last line is not a tensor and T in that sense not a well defined geometric object. Nevertheless it can be a building block of a geometrically well defined object, for example in the definition of the *exterior derivative* on multivector valued forms which we suggested above. Namely, if we define[5]

$$\mathbf{d}K^{(k,k')} \equiv \Big\{ \boldsymbol{o}, K^{(k,k')} \Big\} = \partial_{\boldsymbol{m}} K_{\boldsymbol{m} \dots \boldsymbol{m}}{}^{\boldsymbol{n} \dots \boldsymbol{n}} - (-)^{k-k'} k' \cdot K_{\boldsymbol{m} \dots \boldsymbol{m}}{}^{\boldsymbol{n} \dots \boldsymbol{n} k} p_k \tag{2.32}$$

We get via our extended embedding (2.31) the nice relation [6]

$$\begin{aligned} \imath_{\mathbf{d}K} \rho &= [\mathbf{d}, \imath_K]\, \rho \overset{(B.45)}{=} -(-)^{k-k'} \boldsymbol{\mathcal{L}}_K \rho &(2.33)\\ \text{with} \quad \boldsymbol{\mathcal{L}}_K \rho &= (k')! \binom{r}{k'-1} K_{\boldsymbol{m} \dots \boldsymbol{m}}{}^{l_1 \dots l_{k'}} \partial_{l_{k'}} \rho_{l_{k'-1} \dots l_1 \boldsymbol{m} \dots \boldsymbol{m}} + \\ &\quad - (-)^{k-k'} (k')! \binom{r}{k'} \partial_{\boldsymbol{m}} K_{\boldsymbol{m} \dots \boldsymbol{m}}{}^{l_1 \dots l_{k'}} \rho_{l_{k'} \dots l_1 \boldsymbol{m} \dots \boldsymbol{m}} &(2.34) \end{aligned}$$

[5] This can of course be seen as a BRST differential, which is well known to be the sum of the longitudinal exterior derivate plus the Koszul Tate differential. However, as the constraint surface in our case corresponds to the configuration space (p_k would be the first class constraint generating the BRST-transformation), it is reasonable to regard the BRST differential as a natural extension of the exterior derivative of the configuration space.

[6] The exterior derivative on forms has already earlier (2.9) been seen to coincide with the Poisson bracket with $\boldsymbol{o}$, which can be used to demonstrate (2.33):

$$\begin{aligned} [\mathbf{d}, \imath_K]\, \rho &= \mathbf{d}(\imath_K \rho) - (-)^{|K|} \imath_K (\mathbf{d}\rho) = \{ \boldsymbol{o}, \imath_K \rho \} - (-)^{|K|} \imath_K \{ \boldsymbol{o}, \rho \} \\ &\overset{(2.12)}{=} \partial_{m_1} K_{m_2 \dots m_{k+1}}{}^{n_1 \dots n_{k'}} \boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_{k+1}} \Big\{ \boldsymbol{b}_{n_1}, \Big\{ \boldsymbol{b}_{n_2}, \Big\{ \cdots, \Big\{ \boldsymbol{b}_{n_{k'}}, \rho^{(r)} \Big\}\Big\}\Big\}\Big\} + \\ &\quad + (-)^k k' \cdot K_{m_1 \dots m_k}{}^{n_1 \dots n_{k'}} \boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_k} \Big\{ \underbrace{\{ \boldsymbol{o}, \boldsymbol{b}_{n_1} \}}_{p_{n_1}}, \Big\{ \boldsymbol{b}_{n_2}, \Big\{ \cdots, \Big\{ \boldsymbol{b}_{n_{k'}}, \rho^{(r)} \Big\}\Big\}\Big\}\Big\} \overset{(2.30)}{\underset{(2.32)}{=}} \imath_{\mathbf{d}K} \rho \end{aligned}$$

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$\boldsymbol{\mathcal{L}}_K\rho$ is the natural generalization of the Lie derivative with respect to vectors acting on forms, which is given similarly $\boldsymbol{\mathcal{L}}_v\rho = [\imath_v, \mathbf{d}]\rho$. As $\imath_K$ is a higher order derivative, also $\boldsymbol{\mathcal{L}}_K$ is a higher order derivative. Nevertheless, it will be called *Lie derivative with respect to* K in this paper. Let us again recall this fact: if p_k appears in a combination like $\mathbf{d}K$, there is a well defined geometric meaning and $\mathbf{d}K$ is up to a sign nothing else than the Lie derivative with respect to K, when embedded in the space of differential operators on forms. The commutator with the exterior derivative is a natural differential in the space of differential operators acting on forms, and via the embedding it induces the differential $\mathbf{d}$ on K. It should perhaps be stressed that the above definition of $\mathbf{d}K$ corresponds to an extended action of the exterior derivative which acts also on the basis elements of the tangent space

$$\mathbf{d}(\boldsymbol{\partial}_m) = p_m \tag{2.35}$$

This approach will enable us to give explicit coordinate expressions for the derived bracket of multivector valued forms even in the general case where the result is not a tensor: In the space of differential operators on forms, we have the commutator $[\imath_K, \imath_L]$ and its derived bracket (B.48) $[\imath_K,_{\mathbf{d}}\imath_L] \equiv [[\imath_K, \mathbf{d}], \imath_L]$, while on the space of multivector valued forms we have the algebraic bracket $[K, L]^{\Delta}$ and want to define its derived bracket up to a sign as $[\mathbf{d}K, L]^{\Delta}$. To this end we also have to extend the definition (2.18), (2.19) of $\imath^{(p)}$, which appears in the explicit expression of the algebraic bracket in (2.22) to objects that contain p_k. This is done in a way that the old equations for the algebraic bracket remain formally the same. So let us define[7]

$$\imath^{(p)}_{T(t,t',t'')} \equiv \sum_{q=0}^{p} \binom{t'}{q}\binom{t''}{p-q} T_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} i_1\ldots i_q\, ,\, i_{q+1}\ldots i_p k_1\ldots k_{t''-p+q}} p_{k_1}\cdots p_{k_{t''-p+q}} \times$$
$$\times \frac{\partial^p}{\partial \boldsymbol{c}^{i_1}\ldots\partial \boldsymbol{c}^{i_q}\partial x^{i_{q+1}}\ldots\partial x^{i_p}} \tag{2.36}$$
$$= \frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q} T \frac{\overleftarrow{\partial}^p}{\partial p_{i_p}\ldots\partial p_{i_{q+1}}\partial \boldsymbol{b}_{i_q}\ldots\partial \boldsymbol{b}_{i_1}} \frac{\partial^p}{\partial \boldsymbol{c}^{i_1}\ldots\partial \boldsymbol{c}^{i_q}\partial x^{i_{q+1}}\ldots\partial x^{i_p}} \tag{2.37}$$

For $p = t' + t''$ it coincides with the full interior product (2.31): $\imath^{(t'+t'')}_{T(t,t',t'')} = \imath_{T(t,t',t'')}$. In addition we have with this definition (after some calculation) $\imath^{(p)}_{\mathbf{d}T} = [\mathbf{d}, \imath^{(p)}_T]$ and in particular

$$\imath^{(p)}_{\mathbf{d}K} = [\mathbf{d}, \imath^{(p)}_K] \tag{2.38}$$

and the equations for the algebraic bracket (2.16–2.22) indeed remain formally the same

[7]Note that $\sum_{q=0}^{p}\binom{t'}{q}\binom{t''}{p-q} = \binom{t'+t''}{p}$.

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for objects containing p_m

$$[\imath_{T^{(t,t',t'')}}, \imath_{\tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}}] \equiv \imath_{[T,\tilde{T}]^{\Delta}} \tag{2.39}$$

$$\imath_T \imath_{\tilde{T}} = \sum_{p\geq 0} \imath_{\imath_T^{(p)}\tilde{T}} \tag{2.40}$$

$$[T^{(t,t',t'')}, \tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}]^{\Delta} \equiv \sum_{p\geq 1} \underbrace{\imath_T^{(p)}\tilde{T} - (-)^{(t-t')(\tilde{t}-\tilde{t}')}\imath_{\tilde{T}}^{(p)}T}_{\equiv [T,\tilde{T}]^{\Delta}_{(p)}} \tag{2.41}$$

$$[T,\tilde{T}]^{\Delta}_{(1)} = \left\{T,\tilde{T}\right\} \tag{2.42}$$

which we can again rewrite in terms of "quantum"-operators (2.14) as

$$\left[\hat{T}^{(k,k')}, \hat{\tilde{T}}^{(l,l')}\right] = \sum_{p\geq 1}\left(\frac{\hbar}{i}\right)^p \widehat{\left[T,\tilde{T}\right]^{\Delta}_{(p)}} = \left(\frac{\hbar}{i}\right)\widehat{\left\{T,\tilde{T}\right\}} + \sum_{p\geq 2}\left(\frac{\hbar}{i}\right)^p \widehat{\left[T,\tilde{T}\right]^{\Delta}_{(p)}} \tag{2.43}$$

It should be stressed that — although very useful — $\imath^{(p)}$ is unfortunately NOT a geometric operation any longer in general, in the sense that $\imath_{\mathbf{d}K}^{(p)}L$ and also $\imath_L^{(p)}\mathbf{d}K$ do not have a well defined geometric meaning, although $\mathbf{d}K$ and L have. $\imath_{\mathbf{d}K}\rho$ and $\imath_K^{(p)}L$ are in contrast well defined. $\imath_{\mathbf{d}K}^{(p)}L$, for example, should rather be understood as a building block of a coordinate calculation which combines only in certain combinations (e.g. the bracket $[\,,\,]^{\Delta}$) to s.th. geometrically meaningful.

We are now ready to define the *derived bracket* of the algebraic bracket for multivector valued forms (see footnote 20 on page 44)

$$\left[K^{(k,k')}\boldsymbol{,}L^{(l,l')}\right] \equiv [K,_{\mathbf{d}}L]^{\Delta} \equiv -(-)^{k-k'}[\mathbf{d}K,L]^{\Delta} \tag{2.44}$$

$$= \sum_{p\geq 1} -(-)^{k-k'}\imath_{\mathbf{d}K}^{(p)}L + (-)^{(k+1-k')(l-l')+k-k'}\imath_L^{(p)}\mathbf{d}K \tag{2.45}$$

$$= \sum_{p\geq 1} -(-)^{k-k'}\imath_{\mathbf{d}K}^{(p)}L + (-)^{(k-k'+1)(l-l'+1)}(-)^{l-l'}\imath_{\mathbf{d}L}^{(p)}K$$
$$+(-)^{(k-k')(l-l')+k-k'}\mathbf{d}(\imath_L^{(p)}K) \tag{2.46}$$

The result is geometrical in the sense that after embedding via the interior product it is a well defined operator acting on forms. This is the case, because due to our extended definitions we have for *all* multivector valued forms the relation

$$[[\imath_K,\mathbf{d}],\imath_L] = \imath_{\left[K^{(k,k')},L^{(l,l')}\right]} \tag{2.47}$$

and the lefthand side is certainly a well defined geometric object. A considerable effort went into getting a correct coordinate form for the general derived bracket and for that

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reason, let us quickly have a glance at the final result, although it is kind of ugly:[8]

$$
\begin{aligned}
[K,L] = \sum_{p\geq 1} & -(-)^{k-k'}(-)^{(k'-p)(l-p)} p! \binom{l}{p}\binom{k'}{p} \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} L_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{k+k'l+k'+p+pl+pk'} p! \binom{k}{p}\binom{l'}{p} \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m} k_p\ldots k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\ldots k_p \boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k'l+k'+pl+pk'} p! \binom{k}{p-1}\binom{l'}{p} \partial_l K_{\boldsymbol{m}\ldots\boldsymbol{m} k_{p-1}\ldots k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\ldots k_{p-1} l \boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{(k'-p)(l-p+1)} p! \binom{l}{p-1}\binom{k'}{p} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_{p-1} k} \partial_k L_{l_{p-1}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{(k'-1-p)(l-p)} p!(k'-p) \binom{l}{p}\binom{k'}{p} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p k} L_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} p_k + \\
& -(-)^{k'l+l+pk'+lp} k'\cdot p! \binom{k}{p}\binom{l'}{p} K_{\boldsymbol{m}\ldots\boldsymbol{m} k_p\ldots k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n} k} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\ldots k_p \boldsymbol{n}\ldots\boldsymbol{n}} p_k \qquad (2.48)
\end{aligned}
$$

The result is only a tensor, when both terms with p_k on the righthand side vanish, although the complete expression is in general geometrically well-defined when considered to be a differential operator acting on forms via $\imath_{[K,L]}$ as this equals per definition the well-defined $[[\imath_K, \mathbf{d}], \imath_L]$. The above coordinate form reduces in the appropriate cases to vector Lie-bracket, Schouten-bracket, and (up to a total derivative) to the (Fröhlicher)-Nijenhuis-bracket. If one allows as well sums of a vector and a 1-form, we get the Dorfman bracket, and also the sum of a vector and a general form gives a result without p.

Due to our extended definition of the exterior derivative, we can also define the *derived*

[8] The building blocks are

$$
\begin{aligned}
\imath^{(p)}_{\mathbf{d}K} L = & (-)^{(k'-p)(l-p)} p! \binom{k'}{p}\binom{l}{p} \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} i_1\ldots i_p} L_{i_p\ldots i_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k-k'}(-)^{(k'-1-p)(l-p)} (p+1)! \binom{k'}{p+1}\binom{l}{p} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} i_1\ldots i_p k} L_{i_p\ldots i_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} p_k + \\
& -(-)^{k-k'}(-)^{(k'-p)(l-p+1)} p! \binom{k'}{p}\binom{l}{p-1} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} i_1\ldots i_{p-1} i_p} \partial_{i_p} L_{i_{p-1}\ldots i_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \\
\imath^{(p)}_{L} \mathbf{d}K = & (-)^{(l'-p)(k+1-p)+p} p! \binom{k}{p}\binom{l'}{p} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} k_1\ldots k_p} \partial_{\boldsymbol{m}} K_{k_p\ldots k_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{(l'-p)(k+1-p)} p! \binom{k}{p-1}\binom{l'}{p} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} k_1\ldots k_{p-1} l} \partial_l K_{k_{p-1}\ldots k_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k-k'}(-)^{(l'-p)(k-p)} k'\cdot p! \binom{k}{p}\binom{l'}{p} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} k_1\ldots k_p} K_{k_p\ldots k_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} k} p_k
\end{aligned}
$$

bracket of the big bracket (the Poisson bracket) via

$$\left[K^{(k,k')},_{\mathbf{d}} L^{(l,l')}\right]^{\Delta}_{(1)} \equiv -(-)^{k-k'}\,[\mathbf{d}K, L]^{\Delta}_{(1)} \tag{2.49}$$

$$= -(-)^{k-k'}\,\{\mathbf{d}K, L\} \tag{2.50}$$

which is just the $p=1$ term of the full derived bracket with the explicit coordinate expression

$$\begin{aligned}[K,_{\mathbf{d}} L]^{\Delta}_{(1)} = & -(-)^{k-k'}(-)^{(k'-1)(l-1)} l k' \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1} L_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\ & -(-)^{k+k'l+l} k l' \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m} k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1 \boldsymbol{n}\ldots\boldsymbol{n}} + \\ & -(-)^{k'l+l} l' \partial_l K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l \boldsymbol{n}\ldots\boldsymbol{n}} + \\ & +(-)^{(k'-1)l} k' K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} k} \partial_k L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\ & +(-)^{k'(l-1)} (k'-1) l k' K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1 k} L_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} p_k + \\ & -(-)^{k'l+k'} k' k l' K_{\boldsymbol{m}\ldots\boldsymbol{m} k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n} k} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1 \boldsymbol{n}\ldots\boldsymbol{n}} p_k \end{aligned} \tag{2.51}$$

$$[K \boldsymbol{,} L] = [K,_{\mathbf{d}} L]^{\Delta}_{(1)} - (-)^{k-k'} \sum_{p\geq 2} [\mathbf{d}K, L]^{\Delta}_{(p)} \tag{2.52}$$

Also this bracket takes a very pleasant coordinate form for generalized multivectors (see (C.69) on page 61). In contrast to the full derived bracket, we have no guarantee for this derived bracket to be geometrical itself.

Let us eventually note how one can easily adjust the extended exterior derivative to the twisted case:

$$[\mathbf{d} + H\wedge\,, \imath_K] \equiv \imath_{\mathbf{d}_H K} \tag{2.53}$$

$$\mathbf{d}_H K = \mathbf{d}K + [H, K]^{\Delta} = \mathbf{d}K - (-)^{k-k'} \sum_{p\geq 1} \imath^{(p)}_K H \tag{2.54}$$

with H being an odd closed differential form. It should be stressed that $\mathbf{d} + H\wedge$ is not a differential, but on the operator level its commutator $[\mathbf{d} + H\wedge, \ldots]$ is a differential and thus the above defined $\mathbf{d}_H$ is a differential as well.

2.2 Sigma-Models

A sigma model is a field theory whose fields are embedding functions from a world-volume Σ into a target space M, like in string theory. So far there was no sigma-model explicitly involved into our considerations. One can understand the previous subsection simply as a convenient way to formulate some geometry. The phase space introduced there, however, is like the phase space of a (point particle) sigma model with only one world-volume dimension — the time — which is not showing up in the off-shell phase-space. Let us now naively consider the same setting like before as a sigma model with the coordinates x^m depending on some worldsheet coordinates[9] σ^μ. The resulting model has a very special

[9]The index μ will not include the worldvolume time, when considering the phase space, but it will contain the time in the Lagrangian formalism. As this should be clear from the context, there will be no notational distinction.

field content, because its anticommuting fields $\boldsymbol{c}^m(\sigma)$ have the same index structure as the embedding coordinate $x^m(\sigma)$. In one and two worldvolume-dimensions, $\boldsymbol{c}^m$ can be regarded as worldvolume-fermions, and this will be used in the stringy application in 3.2. In general worldvolume dimensions, $\boldsymbol{c}^m$ could be seen as ghosts, leading to a topological theory. In any case the dimension of the worldvolume will not yet be fixed, as the described mechanism does not depend on it.

A multivector valued form on a C^∞-manifold M can locally be regarded as an analytic function of $x^m, \mathbf{d}x^m \equiv \boldsymbol{c}^m$ and $\boldsymbol{\partial}_m \equiv \boldsymbol{b}_m$

$$K^{(k,k')}(x, \mathbf{d}x, \boldsymbol{\partial}) = K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(x)\mathbf{d}x^{m_1} \wedge \cdots \wedge \mathbf{d}x^{m_k} \wedge \boldsymbol{\partial}_{n_1} \wedge \cdots \wedge \boldsymbol{\partial}_{n_{k'}} \tag{2.55}$$

$$\equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(x)\boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_k}\boldsymbol{b}_{n_1} \cdots \boldsymbol{b}_{n_{k'}}$$

$$= K^{(k,k')}(x, \boldsymbol{c}, \boldsymbol{b}) \tag{2.56}$$

For sigma models, $x^m \to x^m(\sigma), p_m \to p_m(\sigma), \boldsymbol{c}^m \to \boldsymbol{c}^m(\sigma)$ and $\boldsymbol{b}_m \to \boldsymbol{b}_m(\sigma)$ become dependent on the worldvolume variables σ^μ. They are, however, for every σ valid arguments of the function K. Frequently only the worldvolume coordinate σ will then be denoted as new argument of K, which has to be understood in the following sense

$$K^{(k,k')}(\sigma) \equiv K^{(k,k')}\left(x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma)\right)$$

$$= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(x(\sigma)\right) \cdot \boldsymbol{c}^{m_1}(\sigma) \cdots \boldsymbol{c}^{m_k}(\sigma)\boldsymbol{b}_{n_1}(\sigma) \cdots \boldsymbol{b}_{n_{k'}}(\sigma) \tag{2.57}$$

Also functions depending on p_m, like $\mathbf{d}K(x, \boldsymbol{c}, \boldsymbol{b}, p)$ in (2.32), or more general a function $T^{(t,t',t'')}(x, \boldsymbol{c}, \boldsymbol{b}, p)$ as in (2.28) are denoted in this way

$$T^{(t,t',t'')}(\sigma) \equiv T^{(t,t',t'')}\left(x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), p(\sigma)\right) \quad \text{(see (2.28))} \tag{2.58}$$

$$\text{e.g. } \mathbf{d}K(\sigma) \equiv \mathbf{d}K\left(x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), p(\sigma)\right) \quad \text{(see (2.32))} \tag{2.59}$$

$$\text{or } \boldsymbol{o}(\sigma) \equiv \boldsymbol{o}\left(\boldsymbol{c}(\sigma), p(\sigma)\right) = \boldsymbol{c}^m(\sigma)p_m(\sigma) \quad \text{(see (2.8))} \tag{2.60}$$

The expression $\mathbf{d}K(\sigma)$ should *NOT* be mixed up with the worldsheet exterior derivative of K which will be denoted by $\boldsymbol{d}^{\mathrm{w}}K(\sigma)$.[10] Every operation of the previous section, like $\imath_K^{(p)} L$ or the algebraic or derived brackets leads again to functions of $x, \boldsymbol{c}, \boldsymbol{b}$ and sometimes p. Let us use for all of them the notation as above, e.g. for the derived bracket of the big bracket (2.49), (2.51)

$$\left[K^{(k,k')},_{\mathbf{d}} L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma) \equiv \left[K^{(k,k')}\boldsymbol{,} L^{(l,l')}\right]^{(\Delta)}_{(1)}\left(x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), p(\sigma)\right) \tag{2.61}$$

And even $\mathbf{d}x^m = \boldsymbol{c}^m$ and $\mathbf{d}\boldsymbol{b}_m = p_m$ will be seen as a function (identity) of $\boldsymbol{c}^m$ or $\boldsymbol{b}_m$, s.th. we denote

$$\mathbf{d}x^m(\sigma) \equiv \boldsymbol{c}^m(\sigma) \tag{2.62}$$

$$\mathbf{d}\boldsymbol{b}_m(\sigma) \equiv p_m(\sigma) \tag{2.63}$$

[10] It is much better to mix it up with a BRST transformation or with something similar to a worldsheet supersymmetry transformation. We will come to that later in subsection 3.2. To make confusion perfect, it should be added that in contrast it is not completely wrong in subsection 2.5 to mix up the target space exterior derivative with the worldsheet exterior derivative...

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Although $\mathbf{d}$ acts only in the target space on $x, \boldsymbol{b}, \boldsymbol{c}$ and p, the above obviously suggests to introduce a differential – say $\mathbf{s}$ — in the new phase space, which is compatible with the target space differential in the sense

$$\mathbf{s}(x^m(\sigma)) = \mathbf{d}x^m(\sigma) \equiv \boldsymbol{c}^m(\sigma) \tag{2.64}$$

$$\mathbf{s}(\boldsymbol{b}_m(\sigma)) = \mathbf{d}\boldsymbol{b}_m(\sigma) \equiv p_m(\sigma) \tag{2.65}$$

We can generate $\mathbf{s}$ with the Poisson bracket in almost the same way as $\mathbf{d}$ before in (2.8):

$$\boldsymbol{\Omega} \equiv \int_\Sigma d^{d_{\rm w}-1}\sigma \quad \boldsymbol{o}(\sigma) = \int d^{d_{\rm w}-1}\sigma \quad \boldsymbol{c}^m(\sigma)p_m(\sigma), \qquad \mathbf{s}(\ldots) = \{\boldsymbol{\Omega}, \ldots\} \tag{2.66}$$

The Poisson bracket between the conjugate fields gets of course an additional delta function compared to (2.5), (2.6).

$$\left\{p_m(\sigma'), x^n(\sigma)\right\} = \delta^n_m \delta^{d_{\rm w}-1}(\sigma' - \sigma) \tag{2.67}$$

$$\left\{\boldsymbol{b}_m(\sigma'), \boldsymbol{c}^n(\sigma)\right\} = \delta^n_m \delta^{d_{\rm w}-1}(\sigma' - \sigma) \tag{2.68}$$

The first important (but rather trivial) observation is then that for $K(\sigma)$ being a function of $x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma)$ as in (2.57) (and not a functional, which could contain derivatives on or integrations over σ) we have

$$\mathbf{s}(K(\sigma)) = \left(\boldsymbol{c}^m(\sigma)\frac{\partial}{\partial(x^m(\sigma))} + p_m(\sigma)\frac{\partial}{\partial(\boldsymbol{b}_m(\sigma))}\right) K\left(x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma)\right) = \mathbf{d}K(\sigma) \tag{2.69}$$

The same is true for more general objects of the form of T in (2.58). Because of this fact the distinction between $\mathbf{d}$ and $\mathbf{s}$ is not very essential, but in subsection 2.5 the replacement of the arguments as in (2.58) will be different and the distinction very essential in order not to get confused.

The relation between Poisson bracket and big bracket (2.23), (2.42) gets obviously modified by a delta function

$$\left\{K^{(k,k')}(\sigma'), L^{(l,l')}(\sigma)\right\} = \left[K^{(k,k')}, L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma)\, \delta^{d_{\rm w}-1}(\sigma' - \sigma) \tag{2.70}$$

or more general

$$\left\{T^{(t,t',t'')}(\sigma'), \tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}(\sigma)\right\} = \left[T^{(t,t',t'')}, \tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}\right]^{\Delta}_{(1)}(\sigma)\, \delta^{d_{\rm w}-1}(\sigma'-\sigma) \tag{2.71}$$

The relation between the derived bracket (using $\mathbf{s}$) on the lefthand side and the derived bracket (using $\mathbf{d}$) on the righthand side is (omitting the overall sign in the definition of the derived bracket)

$$\begin{aligned}\left\{\mathbf{s}K^{(k,k')}(\sigma'), L^{(l,l')}(\sigma)\right\} &\overset{(2.69)}{=} \left\{\mathbf{d}K^{(k,k')}(\sigma'), L^{(l,l')}(\sigma)\right\} \\ &\overset{(2.71)}{=} \left[\mathbf{d}K^{(k,k')}, L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma)\, \delta^{d_{\rm w}-1}(\sigma' - \sigma)\end{aligned} \tag{2.72}$$

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The worldvolume coordinates σ remain so far more or less only spectators. In the subsection 2.5, the world-volume coordinates play a more active part and already in the following subsection a similar role is taken by an anticommuting extension of the worldsheet.

Before we proceed, it should be stressed that the replacement of $x, \boldsymbol{c}, \boldsymbol{b}$ and p by $x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma)$ and $p(\sigma)$ was just the most naive replacement to do, and it will be a bit extended in the following section until it can serve as a useful tool in an application in 3.2. But in principle, one can replace those variables by any fields with matching index structure and parity (even composite ones) and study the resulting relations between Poisson bracket on the one side and geometric bracket on the other side. Also the differential $\mathbf{s}$ can be replaced for example by the twisted differential or by more general BRST-like transformations. In this way it should be possible to implement other derived brackets, for example those built with the Poisson-Lichnerowicz-differential (see [1]), in a sigma-model description. In 2.5, a different (but also quite canonical) replacement is performed and we will see that the different replacement corresponds to a change of the role of σ and an anticommuting worldvolume coordinate $\boldsymbol{\theta}$ which will be introduced in the following.

2.3 Natural appearance of derived brackets in Poisson brackets of superfields

In the application to worldsheet theories in section 3, there appear superfields, either in the sense of worldsheet supersymmetry or in the sense of de-Rham superfields (see e.g. [24, 19]). Let us view a superfield in general as a method to implement a fermionic transformation of the fields via a shift in a fermionic parameter $\boldsymbol{\theta}$ which can be regarded as fermionic extension of the worldvolume. In our case the fermionic transformation is just the spacetime de-Rham-differential $\mathbf{d}$, or more precisely $\mathbf{s}$, and is not necessarily connected to worldvolume supersymmetry. In fact, in worldvolumes of dimension higher than two, supersymmetry requires more than one fermionic parameter while a single $\boldsymbol{\theta}$ is enough for our purpose to implement $\mathbf{s}$. In two dimensions, however, this single theta can really be seen as a worldsheet fermion (see 3.2). But let us neglect this knowledge for a while, in order to clearly see the mechanism, which will be a bit hidden again, when applied to the supersymmetric case in 3.2.

As just said above, we want to implement with superfields the fermionic transformation $\mathbf{s}$ and not yet a supersymmetry. So let us define in this section a *superfield* as a function of the phase space fields with additional dependence on $\boldsymbol{\theta}$, $Y = Y(x(\sigma), p(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), \boldsymbol{\theta})$, which obeys[11]

$$\mathbf{s}Y(x(\sigma), p(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), \boldsymbol{\theta}) \stackrel{!}{=} \partial_{\boldsymbol{\theta}} Y(x(\sigma), p(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), \boldsymbol{\theta}) \tag{2.73}$$

$$\begin{aligned} \text{with} \quad \mathbf{s}x^m(\sigma) &= \boldsymbol{c}^m(\sigma), \\ \mathbf{s}\boldsymbol{b}_m(\sigma) &= p_m(\sigma) \quad (\mathbf{s}\boldsymbol{\theta} = 0) \end{aligned} \tag{2.74}$$

[11] If this seems unfamiliar, compare with the case of worldsheet supersymmetry, where one introduces a differential operator $\mathrm{Q}_{\boldsymbol{\theta}} \equiv \partial_{\boldsymbol{\theta}} + \boldsymbol{\theta}\partial_\sigma$ and the definition of a superfield is, in contrast to here, $\delta_{\boldsymbol{\varepsilon}} Y \stackrel{!}{=} \boldsymbol{\varepsilon}\mathrm{Q}_{\boldsymbol{\theta}} Y$, where $\delta_{\boldsymbol{\varepsilon}}$ is the supersymmetry transformation of the component fields (compare 3.2).

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With our given field content it is possible to define two basic conjugate[12] superfields Φ^m and $\boldsymbol{S}_m$ which build up a super-phase-space [13]

$$\begin{aligned}
\Phi^m(\sigma,\boldsymbol{\theta}) &\equiv x^m(\sigma)+\boldsymbol{\theta}\boldsymbol{c}^m(\sigma)=x^m(\sigma)+\boldsymbol{\theta}\mathbf{s}x^m(\sigma) \qquad (2.75)\\
\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) &\equiv \boldsymbol{b}_m(\sigma)+\boldsymbol{\theta}p_m(\sigma)=\boldsymbol{b}_m(\sigma)+\boldsymbol{\theta}\mathbf{s}\boldsymbol{b}_m(\sigma)\\
\left\{\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}),\Phi^n(\sigma',\boldsymbol{\theta}')\right\} &= \left\{\boldsymbol{b}_m(\sigma),\boldsymbol{\theta}'\boldsymbol{c}^n(\sigma')\right\}+\boldsymbol{\theta}\left\{p_m(\sigma),x^n(\sigma')\right\}=\underbrace{(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{\equiv\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}\delta(\sigma-\sigma')\delta^n_m
\end{aligned}$$

Φ and $\boldsymbol{S}$ are obviously superfields in the above sense

$$\partial_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta})=\underbrace{\mathbf{s}x^m(\sigma)}_{\boldsymbol{c}^m(\sigma)}\underbrace{+\boldsymbol{\theta}\mathbf{s}\boldsymbol{c}^m(\sigma)}_{=0}=\mathbf{s}\Phi^m(\sigma,\boldsymbol{\theta}) \qquad \partial_{\boldsymbol{\theta}}\boldsymbol{S}_m=\underbrace{\mathbf{s}\boldsymbol{b}_m(\sigma)}_{p_m(\sigma)}\underbrace{+\boldsymbol{\theta}\mathbf{s}p_m(\sigma)}_{0}=\mathbf{s}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) \qquad (2.76)$$

as well as $\mathbf{s}\Phi(\sigma,\boldsymbol{\theta})=\boldsymbol{c}(\sigma)$ and $\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})=p(\sigma)$ are superfields, and every analytic function of those fields will be a superfield again.

---

[12] The superfields Φ and $\boldsymbol{S}$ are conjugate with respect to the following *super-Poisson-bracket*

$$\begin{aligned}
&\left\{F(\sigma',\boldsymbol{\theta}'),G(\sigma,\boldsymbol{\theta})\right\}\equiv\\
&\equiv\int d^{d_{\rm w}-1}\tilde{\sigma}\int d\tilde{\boldsymbol{\theta}}\,\left(\delta F(\sigma',\boldsymbol{\theta}')/\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}G(\sigma,\boldsymbol{\theta})-\delta F(\sigma',\boldsymbol{\theta}')/\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}G(\sigma,\boldsymbol{\theta})\right)\\
&=\int d^{d_{\rm w}-1}\tilde{\sigma}\int d\tilde{\boldsymbol{\theta}}\,\left(\delta F(\sigma',\boldsymbol{\theta}')/\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}G(\sigma,\boldsymbol{\theta})-(-)^{FG}\delta G(\sigma',\boldsymbol{\theta}')/\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}F(\sigma,\boldsymbol{\theta})\right)
\end{aligned}$$

which, however, boils down to taking the ordinary graded Poisson bracket between the component fields (as can be seen in (2.75)). The *functional derivatives* from the left and from the right are defined as usual via

$$\delta_S A\equiv\int d^{d_{\rm w}-1}\tilde{\sigma}\int d\tilde{\theta}\quad\delta A/\delta S_k(\tilde{\sigma},\tilde{\theta})\cdot\delta S_k(\tilde{\sigma},\tilde{\theta})\equiv\int d^{d_{\rm w}-1}\tilde{\sigma}\int d\tilde{\theta}\quad\delta S_k(\tilde{\sigma},\tilde{\theta})\cdot\frac{\delta}{\delta S_k(\tilde{\sigma},\tilde{\theta})}A$$

and similarly for Φ, which leads to

$$\begin{aligned}
\frac{\delta}{\delta\boldsymbol{S}_m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}\boldsymbol{S}_n(\sigma,\boldsymbol{\theta}) &= \delta^m_n(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})\delta^{d_{\rm w}-1}(\sigma-\tilde{\sigma})=-\delta\boldsymbol{S}_n(\sigma,\boldsymbol{\theta})/\boldsymbol{S}_m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\\
\frac{\delta}{\delta\Phi^m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}\Phi^n(\sigma,\boldsymbol{\theta}) &= \delta^n_m(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta})\delta^{d_{\rm w}-1}(\sigma-\tilde{\sigma})=\delta\Phi^n(\sigma,\boldsymbol{\theta})/\delta\Phi^m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})
\end{aligned}$$

The functional derivatives can also be split in those with respect to the component fields

$$\frac{\delta}{\delta\boldsymbol{S}_m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}=\frac{\delta}{\delta p_m(\tilde{\sigma})}-\tilde{\boldsymbol{\theta}}\frac{\delta}{\delta\boldsymbol{b}_m(\tilde{\sigma})},\qquad\frac{\delta}{\delta\Phi^m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}=\frac{\delta}{\delta\boldsymbol{c}^m(\tilde{\sigma})}+\tilde{\boldsymbol{\theta}}\frac{\delta}{\delta x^m(\tilde{\sigma})}$$

[13] For Grassmann variables $\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})=\boldsymbol{\theta}'-\boldsymbol{\theta}$ in the following sense

$$\begin{aligned}
\int\mathbf{d}\boldsymbol{\theta}'(\boldsymbol{\theta}'-\boldsymbol{\theta})F(\boldsymbol{\theta}') &= \int\mathbf{d}\boldsymbol{\theta}'(\boldsymbol{\theta}'-\boldsymbol{\theta})\left(F(\boldsymbol{\theta})+(\boldsymbol{\theta}'-\boldsymbol{\theta})\partial_{\boldsymbol{\theta}}F(\boldsymbol{\theta})\right)=\\
&= \int\mathbf{d}\boldsymbol{\theta}'\quad\boldsymbol{\theta}'F(\boldsymbol{\theta})-\boldsymbol{\theta}'\boldsymbol{\theta}\partial_{\boldsymbol{\theta}}F(\boldsymbol{\theta})-\boldsymbol{\theta}\boldsymbol{\theta}'\partial_{\boldsymbol{\theta}}F(\boldsymbol{\theta})=F(\boldsymbol{\theta})
\end{aligned}$$

We have as usual

$$\boldsymbol{\theta}\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})=\boldsymbol{\theta}(\boldsymbol{\theta}'-\boldsymbol{\theta})=\boldsymbol{\theta}\boldsymbol{\theta}'=\boldsymbol{\theta}'(\boldsymbol{\theta}'-\boldsymbol{\theta})=\boldsymbol{\theta}'\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})$$

Pay attention to the antisymmetry

$$\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\;=\;-\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')$$

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We will convince ourselves in this subsection that in the Poisson brackets of general superfields, the derived brackets come with the complete δ-function (of σ and $\boldsymbol{\theta}$) while the corresponding algebraic brackets come with a derivative of the delta-function. The introduction of worldsheet coordinates σ was not yet really necessary for this discussion, but it will be useful for the comparison with the subsequent subsection. Indeed, we do not specify the dimension $d_{\rm w}$ of the worldsheet yet. An argument sigma is representing several worldsheet coordinates σ^μ. It should be stressed again that the differential $\mathbf{d}$ should *NOT* be mixed up with the worldsheet exterior derivative $\boldsymbol{d}^{\rm w}$, which does not show up in this subsection.

Similar as in 2.2, equations (2.57)–(2.63),we will view all geometric objects as functions of local coordinates and replace the arguments not by phase space fields but by the just defined super-phase fields which reduces for $\boldsymbol{\theta} = 0$ to the previous case.

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) \equiv T^{(t,t',t'')}\left(\Phi(\sigma,\boldsymbol{\theta}),\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \overset{\boldsymbol{\theta}=0}{=} T^{(t,t',t'')}(\sigma) \quad \text{(see (2.58))} \tag{2.77}$$

For example for a multivector valued form we write

$$\begin{aligned} K^{(k,k')}(\sigma,\boldsymbol{\theta}) &\equiv K^{(k,k')}\big(\Phi^m(\sigma,\boldsymbol{\theta}),\underbrace{\mathbf{s}\Phi^m(\sigma,\boldsymbol{\theta})}_{\boldsymbol{c}^m(\sigma)},\boldsymbol{S}_m(\sigma,\boldsymbol{\theta})\big) \\ &= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(\Phi(\sigma,\boldsymbol{\theta})\right)\underbrace{\mathbf{s}\Phi^{m_1}(\sigma,\boldsymbol{\theta})}_{\boldsymbol{c}^{m_1}(\sigma)}\ldots\mathbf{s}\Phi^{m_k}(\sigma,\boldsymbol{\theta})\boldsymbol{S}_{n_1}(\sigma,\boldsymbol{\theta})\ldots\boldsymbol{S}_{n_{k'}}(\sigma,\boldsymbol{\theta}) \\ &\underset{(2.57)}{\overset{\boldsymbol{\theta}=0}{=}} K^{(k,k')}(\sigma) \end{aligned} \tag{2.78}$$

Likewise for all the other examples of 2.2:

$$\text{e.g. } \mathbf{d}K(\sigma,\boldsymbol{\theta}) \equiv \mathbf{d}K\left(\Phi(\sigma,\boldsymbol{\theta}),\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \tag{2.79}$$

$$\text{or } \boldsymbol{o}(\sigma,\boldsymbol{\theta}) \equiv \boldsymbol{o}\left(\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}),\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) = \boldsymbol{c}^m(\sigma)p_m(\sigma) = \boldsymbol{o}(\sigma) \tag{2.80}$$

$$\Big[K^{(k,k')},_{\mathbf{d}}L^{(l,l')}\Big]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \equiv \underbrace{\Big[K^{(k,k')},L^{(l,l')}\Big]^{(\Delta)}_{(1)}\left(\Phi(\sigma,\boldsymbol{\theta}),\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right)}_{\underset{(2.61)}{\overset{\boldsymbol{\theta}=0}{=}}\big[K^{(k,k')},L^{(l,l')}\big]^{(\Delta)}_{(1)}(\sigma)} \tag{2.81}$$

$$\mathbf{d}x^m(\sigma,\boldsymbol{\theta}) \equiv \mathbf{s}\Phi^m(\sigma,\boldsymbol{\theta}) = \boldsymbol{c}^m(\sigma) \tag{2.82}$$

$$\mathbf{d}\boldsymbol{b}_m(\sigma,\boldsymbol{\theta}) \equiv \mathbf{s}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = p_m(\sigma) \tag{2.83}$$

For functions of the type $T^{(t,t',t'')}(\sigma,\boldsymbol{\theta})$ we clearly have

$$\mathbf{d}T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) = \mathbf{s}\Big(T^{(t,t',t'')}(\sigma,\boldsymbol{\theta})\Big) \tag{2.84}$$

$$\text{in particular} \quad \mathbf{d}K^{(k,k')}(\sigma,\boldsymbol{\theta}) = \mathbf{s}\Big(K^{(k,k')}(\sigma,\boldsymbol{\theta})\Big) \tag{2.85}$$

As all those analytic functions of the basic superfields are superfields (in the sense of 2.73) themselves, $\partial_{\boldsymbol{\theta}}$ can be replaced by $\mathbf{s}$ in a $\boldsymbol{\theta}$-expansion, so that we get the important relation

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) = T^{(t,t',t'')}(\sigma) + \boldsymbol{\theta}\mathbf{d}T^{(t,t',t'')}(\sigma) \tag{2.86}$$

$$K^{(k,k')}(\sigma,\boldsymbol{\theta}) = K^{(k,k')}(\sigma) + \boldsymbol{\theta}\mathbf{d}K^{(k,k')}(\sigma) \tag{2.87}$$

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This also implies that $\mathbf{d}T(\sigma,\boldsymbol{\theta})$ and in particular $\mathbf{d}K(\sigma,\boldsymbol{\theta})$ do actually not depend on $\boldsymbol{\theta}$:

$$\mathbf{d}K^{(k,k')}(\sigma,\boldsymbol{\theta}) = \mathbf{d}K^{(k,k')}(\sigma) \tag{2.88}$$

Now comes the nice part:

Proposition 1. For all multivector valued forms $K^{(k,k')}, L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$ as in (2.10), the following equation holds for the corresponding superfields (2.78)

$$\begin{aligned}&\{K^{(k,k')}(\sigma',\boldsymbol{\theta}'),L^{(l,l')}(\sigma,\boldsymbol{\theta})\}\\ &\quad= \delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')\cdot\underbrace{[\mathbf{d}K,L]^{\Delta}_{(1)}}_{-(-)^{k-k'}[K,_{\mathbf{d}}L]^{\Delta}_{(1)}}(\sigma,\boldsymbol{\theta}) + \underbrace{\partial_{\boldsymbol{\theta}}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{=1}\delta(\sigma-\sigma')[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})\end{aligned} \tag{2.89}$$

where $[K,L]^{\Delta}_{(1)}$ is the big bracket (2.23) (Buttin's algebraic bracket, which was previously just the Poisson bracket, being true now up to a $\delta(\sigma-\sigma')$ only after setting $\boldsymbol{\theta}=\boldsymbol{\theta}'$) and $[K,_{\mathbf{d}}L]^{\Delta}_{(1)}$ is the derived bracket of the big bracket (2.49).

Proof Using (2.87), we can simply plug $K(\sigma',\boldsymbol{\theta}') = K(\sigma')+\boldsymbol{\theta}'\mathbf{d}K(\sigma')$ and $L(\sigma,\boldsymbol{\theta}) = L(\sigma)+\boldsymbol{\theta}\mathbf{d}L(\sigma)$ into the lefthand side:

$$\begin{aligned}\big\{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\big\} &= \big\{K(\sigma'),L(\sigma)\big\}+\boldsymbol{\theta}'\big\{\mathbf{d}K(\sigma'),L(\sigma)\big\}+(-)^{k-k'}\boldsymbol{\theta}\big\{K(\sigma'),\mathbf{d}L(\sigma)\big\}\\ &\quad+(-)^{k-k'}\boldsymbol{\theta}\boldsymbol{\theta}'\big\{\mathbf{d}K(\sigma'),\mathbf{d}L(\sigma)\big\} &(2.90)\\ &= \big\{K(\sigma'),L(\sigma)\big\}+(\boldsymbol{\theta}'-\boldsymbol{\theta})\big\{\mathbf{d}K(\sigma'),L(\sigma)\big\}\\ &\quad+\boldsymbol{\theta}\mathbf{d}\big\{K(\sigma'),L(\sigma)\big\}-\boldsymbol{\theta}\boldsymbol{\theta}'\mathbf{d}\big\{\mathbf{d}K(\sigma'),L(\sigma)\big\} &(2.91)\\ &\overset{(2.23)}{=} \delta(\sigma-\sigma')\Big([K,L]^{\Delta}_{(1)}(\sigma)+\boldsymbol{\theta}\mathbf{d}[K,L]^{\Delta}_{(1)}(\sigma)\Big)+\\ &\quad+(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')\Big([\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma)+\boldsymbol{\theta}\mathbf{d}[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma)\Big) &(2.92)\\ &\overset{(2.86)}{=} \delta(\sigma-\sigma')[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})+(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \quad\square\end{aligned}$$

There is yet another way to see that the bracket at the plain delta functions is the derived bracket of the one at the derivative of the delta-function, which will be useful later: Denote the coefficients in front of the delta-functions by $A(K,L)$ and $B(K,L)$:

$$\big\{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\big\} = A(K,L)\cdot\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')+B(K,L)(\sigma,\boldsymbol{\theta})\underbrace{\partial_{\boldsymbol{\theta}}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{=1}\delta(\sigma-\sigma') \tag{2.93}$$

In order to hit the delta-functions, it is enough to integrate over a patch $U(\sigma)$ containing

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the point parametrized by σ. We can thus extract A and B via

$$A(K,L)(\sigma,\boldsymbol{\theta}) = \int \mathbf{d}\boldsymbol{\theta}' \int_{U(\sigma)} d^{d_w-1}\sigma' \left\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\right\} = \tag{2.94}$$

$$= \int \mathbf{d}\boldsymbol{\theta}' \int d^{d_w-1}\sigma' \left\{K(\sigma') + \boldsymbol{\theta}'\mathbf{d}K(\sigma'), L(\sigma,\boldsymbol{\theta})\right\} = \tag{2.95}$$

$$= \int d^{d_w-1}\sigma' \{ \underbrace{\mathbf{d}K(\sigma')}_{\overset{(2.88)}{=} \mathbf{d}K(\sigma',\boldsymbol{\theta})}, L(\sigma,\boldsymbol{\theta})\} \tag{2.96}$$

$$B(K,L)(\sigma,\boldsymbol{\theta}) = \int \mathbf{d}\boldsymbol{\theta}' \int_{U(\sigma)} d^{d_w-1}\sigma' (\boldsymbol{\theta}'-\boldsymbol{\theta}) \left\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\right\} = \tag{2.97}$$

$$= \int d^{d_w-1}\sigma' \left\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\right\} |_{\boldsymbol{\theta}'=\boldsymbol{\theta}} \tag{2.98}$$

$$\Rightarrow A(K,L) = B(\mathbf{d}K, L) \tag{2.99}$$

It is thus enough to collect in a direct calculation the terms at the derivative of the delta-function and verify that it leads to the big bracket. □

2.4 Comment on the quantum case

In (2.14) the embedding via the interior product into the space of operators acting on forms was interpreted as quantization . In the presence of world-volume dimensions, the partial derivative as Schroedinger representation for conjugate momenta is no longer appropriate and one has to switch to the functional derivative. Remember

$$\Phi^m(\sigma,\boldsymbol{\theta}) = x^m(\sigma) + \boldsymbol{\theta}\boldsymbol{c}^m(\sigma), \qquad \mathbf{d}\Phi^m(\sigma,\boldsymbol{\theta}) = \boldsymbol{c}^m(\sigma) = \mathbf{d}\Phi(\sigma) \tag{2.100}$$

$$\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = \boldsymbol{b}_m(\sigma) + \boldsymbol{\theta}p_m(\sigma), \qquad \mathbf{d}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = p_m(\sigma) = \mathbf{d}\boldsymbol{S}(\sigma) \tag{2.101}$$

The quantization of the superfields in the Schroedinger representation (conjugate momenta as super functional derivatives) is consistent with the quantization of the component fields (see also footnote 12)

$$\hat{\boldsymbol{S}}_m(\sigma,\boldsymbol{\theta}) \equiv \frac{\hbar}{i}\frac{\delta}{\delta\Phi^m(\sigma,\boldsymbol{\theta})} = \frac{\hbar}{i}\frac{\delta}{\delta\boldsymbol{c}^m(\sigma)} + \boldsymbol{\theta}\frac{\hbar}{i}\frac{\delta}{\delta x^m(\sigma)} \tag{2.102}$$

$$\Rightarrow \left[\hat{\boldsymbol{S}}_m(\sigma,\boldsymbol{\theta}), \hat{\Phi}^n(\sigma',\boldsymbol{\theta}')\right] = \frac{\hbar}{i}\left(\frac{\delta}{\delta\boldsymbol{c}^m(\sigma)} + \boldsymbol{\theta}\frac{\delta}{\delta x^m(\sigma)}\right)\left(x^n(\sigma') + \boldsymbol{\theta}'\boldsymbol{c}^n(\sigma')\right) = \tag{2.103}$$

$$= \frac{\hbar}{i}\delta^n_m \left(\boldsymbol{\theta}-\boldsymbol{\theta}'\right)\delta(\sigma-\sigma') \tag{2.104}$$

The quantization of a multivector valued form, containing several operators $\hat{\boldsymbol{S}}$ at the same worldvolume-point, however, leads to powers of delta functions with the same argument when acting on some wave functional. This is the usual problem in quantum field theory and requires a model dependent regularization and renormalization. We will stay model independent here and therefore will not treat the quantum case for a present worldvolume coordinate σ. Nevertheless it is instructive to study it for absent σ, but keeping $\boldsymbol{\theta}$ and considering "worldline-superfields" of the form

$$\Phi^m(\boldsymbol{\theta}) = x^m + \boldsymbol{\theta}\boldsymbol{c}^m, \qquad \mathbf{d}\Phi^m(\boldsymbol{\theta}) = \boldsymbol{c}^m \tag{2.105}$$

$$\boldsymbol{S}_m(\boldsymbol{\theta}) = \boldsymbol{b}_m + \boldsymbol{\theta}p_m, \qquad \mathbf{d}\boldsymbol{S}_m(\boldsymbol{\theta}) = p_m \tag{2.106}$$

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Quantum operator and commutator simplify to

$$\hat{\boldsymbol{S}}_m(\boldsymbol{\theta}) \equiv \frac{\hbar}{i}\frac{\delta}{\delta\Phi^m(\boldsymbol{\theta})} = \frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{c}^m} + \boldsymbol{\theta}\frac{\hbar}{i}\frac{\partial}{\partial x^m} \tag{2.107}$$

$$\Rightarrow \left[\hat{\boldsymbol{S}}_m(\boldsymbol{\theta}), \hat{\Phi}^n(\boldsymbol{\theta}')\right] = \frac{\hbar}{i}\delta^n_m\left(\boldsymbol{\theta}-\boldsymbol{\theta}'\right) \tag{2.108}$$

$$\left[\hat{\boldsymbol{S}}_m(\boldsymbol{\theta}), \widehat{\mathbf{d}\Phi}^n(\boldsymbol{\theta}')\right] = \frac{\hbar}{i}\delta^n_m \tag{2.109}$$

In contrast to σ, products of $\boldsymbol{\theta}$-delta functions are no problem.

The important relation $K(\boldsymbol{\theta}) = K + \boldsymbol{\theta}\mathbf{d}K$ (2.87) can be extended to the quantum case as seen when acting on some r-form.

$$\imath_{K^{(k,k')}}\rho^{(r)}(\boldsymbol{\theta}) \overset{(2.86)}{=} \imath_K\rho + \boldsymbol{\theta}\mathbf{d}(\imath_K\rho) \tag{2.110}$$

$$\overset{(2.33)}{=} \imath_K\rho + \boldsymbol{\theta}\left(\imath_{\mathbf{d}K}\rho + (-)^{k-k'}\imath_K\mathbf{d}\rho\right) = \imath_K(\boldsymbol{\theta})\left(\rho(\boldsymbol{\theta})\right) \tag{2.111}$$

$$\text{with } \imath_K(\boldsymbol{\theta}) \equiv \imath_K + \boldsymbol{\theta}\left[\mathbf{d}, \imath_K\right] \tag{2.112}$$

In that sense we have (remember $\hat{K} = \left(\frac{\hbar}{i}\right)^{k'}\imath_K$)

$$\hat{K}^{(k,k')}(\boldsymbol{\theta}) = \hat{K}^{(k,k')} + \boldsymbol{\theta}\widehat{\mathbf{d}K} \tag{2.113}$$

$$\text{with } \widehat{\mathbf{d}K} \overset{(2.33)}{=} \left[\mathbf{d}, \hat{K}\right] \tag{2.114}$$

where the explicit form of this quantized multivector valued form reads

$$\hat{K}^{(k,k')}(\boldsymbol{\theta}) \equiv \left(\frac{\hbar}{i}\right)^{k'} K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(\Phi(\boldsymbol{\theta})\right)\underbrace{\mathbf{d}\Phi^{m_1}(\boldsymbol{\theta})}_{\boldsymbol{c}^{m_1}}\ldots\mathbf{d}\Phi^{m_k}(\boldsymbol{\theta})\frac{\delta}{\delta\Phi^{n_1}(\boldsymbol{\theta})}\cdots\frac{\delta}{\delta\Phi^{n_{k'}}(\boldsymbol{\theta})} \tag{2.115}$$

In the derivation of (2.112), $\imath_K$ and ρ both were evaluated at the same $\boldsymbol{\theta}$. Let us eventually consider the general case:

$$\hat{K}^{(k,k')}(\boldsymbol{\theta}')\rho^{(r)}(\boldsymbol{\theta}) = \left(\hat{K} + \boldsymbol{\theta}'\widehat{\mathbf{d}K}\right)\left(\rho + \boldsymbol{\theta}\mathbf{d}\rho\right) \tag{2.116}$$

$$= \hat{K}\rho + \boldsymbol{\theta}'\widehat{\mathbf{d}K}\rho + (-)^{k-k'}\boldsymbol{\theta}\hat{K}\mathbf{d}\rho + (-)^{k-k'}\boldsymbol{\theta}\boldsymbol{\theta}'\widehat{\mathbf{d}K}\mathbf{d}\rho \tag{2.117}$$

$$= \hat{K}\rho + \boldsymbol{\theta}\mathbf{d}\left(\hat{K}\rho\right) + (\boldsymbol{\theta}'-\boldsymbol{\theta})\left(\widehat{\mathbf{d}K}\rho + \boldsymbol{\theta}\mathbf{d}\left(\widehat{\mathbf{d}K}\rho\right)\right) \tag{2.118}$$

The relation between quantum operators acting on forms and the interior product therefore becomes modified in comparison to (2.14) and reads

$$\hat{K}^{(k,k')}(\boldsymbol{\theta}')\rho^{(r)}(\boldsymbol{\theta}) = \left(\frac{\hbar}{i}\right)^{k'}\left(\imath_K\rho(\boldsymbol{\theta}) + (\boldsymbol{\theta}'-\boldsymbol{\theta})\underbrace{\imath_{\mathbf{d}K}\rho(\boldsymbol{\theta})}_{(-)^{k-k'}\boldsymbol{\mathcal{L}}_K\rho}\right) \tag{2.119}$$

Proposition 2. For all multivector valued forms $K^{(k,k')}, L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$ as in (2.10), the following equations holds for the corresponding quantized worldline-superfields (2.115) $\hat{K}(\boldsymbol{\theta})$

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and $\hat{L}(\boldsymbol{\theta})$:

$$\left[\hat{K}^{(k,k')}(\boldsymbol{\theta}'),\hat{L}^{(l,l')}(\boldsymbol{\theta})\right] = \sum_{p\geq 1}\left(\frac{\hbar}{i}\right)^{p}\left(\underbrace{\partial_{\boldsymbol{\theta}}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{=1}\widehat{[K,L]^{\Delta}_{(p)}}(\boldsymbol{\theta})+\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\widehat{[\mathbf{d}K,L]^{\Delta}_{(p)}}(\boldsymbol{\theta})\right)$$

$$\begin{aligned}\left[\hat{K}^{(k,k')}(\boldsymbol{\theta}'),\hat{L}^{(l,l')}(\boldsymbol{\theta})\right]\rho(\tilde{\boldsymbol{\theta}}) = \left(\frac{\hbar}{i}\right)^{k'+l'}&\Big(\imath_{[K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}})+\delta(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})\imath_{\mathbf{d}[K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}}) \\ &+\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\left(\imath_{[\mathbf{d}K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}})+\delta(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})\imath_{\mathbf{d}[\mathbf{d}K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}})\right)\Big)\end{aligned} \quad (2.120)$$

Again the algebraic bracket (B.42) comes with the derivative of the delta function while the derived bracket (2.44) comes with the plain delta functions. But this time the algebraic bracket is not only the big bracket $[\,,\,]^{\Delta}_{(1)}$, but the full one.

Proof Let us just plug in (2.113) into the lefthand side:

$$\begin{aligned}[\hat{K}(\boldsymbol{\theta}'),\hat{L}(\boldsymbol{\theta})] &= [\hat{K}+\boldsymbol{\theta}'\widehat{\mathbf{d}K}\,,\,\hat{L}+\boldsymbol{\theta}\widehat{\mathbf{d}L}] \\ &= [\hat{K},\hat{L}]+\boldsymbol{\theta}'[\widehat{\mathbf{d}K}\,,\,\hat{L}]+(-)^{k-k'}\boldsymbol{\theta}[\hat{K}\,,\,\widehat{\mathbf{d}L}]-(-)^{k-k'}\boldsymbol{\theta}'\boldsymbol{\theta}[\widehat{\mathbf{d}K}\,,\,\widehat{\mathbf{d}L}] \\ &\overset{(2.114)}{=} [\hat{K},\hat{L}]+\boldsymbol{\theta}\left[\mathbf{d},[\hat{K}\,,\,\hat{L}]\right]+(\boldsymbol{\theta}'-\boldsymbol{\theta})\left([\widehat{\mathbf{d}K}\,,\,\hat{L}]+\boldsymbol{\theta}\left[\mathbf{d},[\widehat{\mathbf{d}K}\,,\,\hat{L}]\right]\right) \\ &= [\hat{K},\hat{L}](\boldsymbol{\theta})+(\boldsymbol{\theta}'-\boldsymbol{\theta})[\widehat{\mathbf{d}K}\,,\,\hat{L}]\end{aligned} \quad (2.121)$$

Remember now the algebraic bracket (B.41)

$$[\imath_{K^{(k,k')}},\imath_{L^{(l,,l')}}] = \imath_{[K,L]^{\Delta}} = \sum_{p\geq 1}\imath_{[K,L]^{\Delta}_{(p)}} \quad (2.122)$$

$$\text{with } [K,L]^{\Delta}_{(p)} \equiv \imath^{(p)}_{K}L-(-)^{(k-k')(l-l')}\imath^{(p)}_{L}K \quad (2.123)$$

or likewise written in terms of $\hat{K}$ and $\hat{L}$

$$[\hat{K}^{(k,k')},\hat{L}^{(l,l')}] = \sum_{p\geq 1}\left(\frac{\hbar}{i}\right)^{p}\widehat{[K,L]^{\Delta}_{(p)}} \quad (2.124)$$

Due to (2.43) we have exactly the same equation for $[\widehat{\mathbf{d}K}\,,\,\hat{L}]$. Plugging this back into (2.121) completes the proof of (2.120). The second equation in the proposition is just a simple rewriting, when acting on a form, which enables to combine the p-th terms of algebraic and derived bracket to the complete ones. □

2.5 Analogy for the antibracket

In the previous subsection the target space exterior derivative $\mathbf{d}$ (realized in the σ-model phase-space by $\mathbf{s}$) was induced by the the derivative $\partial_{\boldsymbol{\theta}}$ with respect to the anticommuting coordinate. But thinking of the pullback of forms in the target space to worldvolume-forms, $\mathbf{d}$ can of course also be induced to some extend by the derivative with respect to the bosonic worldvolume coordinates σ^{μ} (including the time, because we are in the Lagrangian formalism now) or better by the worldvolume exterior derivative $\boldsymbol{d}^{\mathrm{w}}$. To this end, however,

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we have to make a different identification of the basis elements in tangent- and cotangent-space of the target space with the fields on the worldvolume than before, namely[14]

$$\mathbf{d}x^m \to \boldsymbol{d}^{\mathrm{w}} x^m(\sigma) = \boldsymbol{d}^{\mathrm{w}}\sigma^\mu \partial_\mu x^m(\sigma), \qquad \boldsymbol{\partial}_m \to \boldsymbol{x}^+_m(\sigma) \tag{2.125}$$

where $\boldsymbol{x}^+_m$ is the antifield of x^m, i.e. the conjugate field to x^m with respect to the antibracket.[15] Let us rename

$$\boldsymbol{\theta}^\mu \equiv \boldsymbol{d}^{\mathrm{w}}\sigma^\mu \tag{2.126}$$

For a target space r-form

$$\rho^{(r)}(x^m, \mathbf{d}x^m) \equiv \rho_{m_1\dots m_r}(x)\mathbf{d}x^{m_1}\cdots \mathbf{d}x^{m_r} \tag{2.127}$$

we define (in analogy to (2.78), but indicating that we allow in the beginning only a variation in σ)

$$\rho^{(r)}_{\boldsymbol{\theta}}(\sigma) \equiv \rho^{(r)}(x^m(\sigma), \boldsymbol{d}^{\mathrm{w}}x^m(\sigma)) = \rho_{m_1\dots m_r}(x(\sigma))\boldsymbol{d}^{\mathrm{w}}x^{m_1}(\sigma)\cdots \boldsymbol{d}^{\mathrm{w}}x^{m_r}(\sigma) \tag{2.128}$$

Attention: this vanishes identically for $r > d_{\mathrm{w}}$ (worldvolume dimension).

The worldvolume exterior derivative then induces the target space exterior derivative in the following sense

$$\boldsymbol{d}^{\mathrm{w}}\rho^{(r)}_{\boldsymbol{\theta}}(\sigma) = (\mathbf{d}\rho^{(r)})_{\boldsymbol{\theta}}(\sigma) \tag{2.129}$$

[14]This identification resembles the one in [2] with $\boldsymbol{\partial}_m \to p_m(z)$ and $\mathbf{d}x^m \to \partial x^m(z)$, or $\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_p} \to \epsilon^{\mu_1\cdots\mu_p}\partial_{\mu_1}x^{m_1}(\sigma)\cdots\partial_{\mu_p}x^{m_p}(\sigma)$ in [4]. It is observed in [2] that the Poisson bracket induces the Dorfman bracket between sums of vectors and 1-forms (in generalized geometry) and in [4] more generally that the Poisson-bracket for the p-brane induces the corresponding bracket between sums of vectors and p-forms (which is called, Vinogradov bracket in [4]). As ∂x^m and p_m are commuting phase space variables, higher rank tensors would automatically be symmetrized (only volume forms, i.e. p-forms on a p-brane, can be implemented, using the epsilon-tensor). Symmetrized tensors and brackets inbetween (e.g. the Schouten bracket for symmetric multivectors) make sense and one could transfer the present analysis to this setting, but in general a natural exterior derivative is missing. Therefore the analysis for the above identifications is done in the antifield-formalism. The appearing derived brackets will also contain the Dorfman bracket and the corresponding bracket for sums of vectors and p-forms and in that sense the present approach is a generalization of the observations above.

[15]The antibracket looks similar to the Poisson-bracket, but their conjugate fields have opposite parity, which leads to a different symmetry (namely that of a Lie-bracket of degree +1 (or -1), i.e. the one in a Gerstenhaber algebra or Schouten-algebra, see footnote 18)

$$\begin{aligned}
(A,B) &\equiv \int d\tilde\sigma^{d_{\mathrm{w}}} \left(\delta A/\boldsymbol{x}^+_k(\tilde\sigma)\frac{\delta}{\delta x^k(\tilde\sigma)}B - \delta A/\delta x^k(\tilde\sigma)\frac{\delta}{\delta \boldsymbol{x}^+_k(\tilde\sigma)}B\right)\\
&= \int d\tilde\sigma^{d_{\mathrm{w}}} \left(\delta A/\boldsymbol{x}^+_k(\tilde\sigma)\frac{\delta}{\delta x^k(\tilde\sigma)}B - (-)^{(A+1)(B+1)}\delta B/\boldsymbol{x}^+_k(\tilde\sigma)\frac{\delta}{\delta x^k(\tilde\sigma)}A\right)\\
(A,B) &= -(-)^{(A+1)(B+1)}(B,A)\\
(\boldsymbol{x}^+_m(\sigma),B) &= \frac{\delta}{\delta x^m(\sigma)}B = -\left(B,\boldsymbol{x}^+_m(\sigma)\right)\\
(x^m(\sigma),B) &= -\frac{\delta}{\delta \boldsymbol{x}^+_m(\sigma)}B = (-)^B(B,x^m(\sigma))
\end{aligned}$$

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Again both sides vanish identically for now $r+1 > d_{\rm w}$, which means that in this way one can calculate with target space fields of form degree not bigger than the worldvolume dimension. If we want to have the same relation for $K^{(k,k')}_{\boldsymbol{\theta}}(\sigma)$ (defined in the analogous way), we have to extend the identification in (2.125) by

$$p_m \to \boldsymbol{d}^{\rm w}\boldsymbol{x}^+_m(\sigma) \tag{2.130}$$

and get

$$\boldsymbol{d}^{\rm w} K^{(k,k')}_{\boldsymbol{\theta}}(\sigma) = (\mathbf{d}K^{(k,k')})_{\boldsymbol{\theta}}(\sigma) \tag{2.131}$$

with

$$K^{(k,k')}_{\boldsymbol{\theta}}(\sigma) \equiv K^{(k,k')}\left(x^m(\sigma), \boldsymbol{d}^{\rm w}x^m(\sigma), \boldsymbol{x}^+_m(\sigma)\right) \tag{2.132}$$

$$(\mathbf{d}K^{(k,k')})_{\boldsymbol{\theta}}(\sigma) \equiv \mathbf{d}K^{(k,k')}\left(x^m(\sigma), \boldsymbol{d}^{\rm w}x^m(\sigma), \boldsymbol{x}^+_m(\sigma), \boldsymbol{d}^{\rm w}\boldsymbol{x}^+_m(\sigma)\right) \tag{2.133}$$

The analysis is thus very similar to that of the previous section.

Proposition 3a. For all multivector valued forms $K^{(k,k')}, L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$, the following equation holds for the corresponding sigma-model realizations (2.132), (2.133)

$$(K_{\boldsymbol{\theta}}(\sigma'),L_{\boldsymbol{\theta}}(\sigma)) = \big(\underbrace{[K,_{\mathbf{d}}L]^{\Delta}_{(1)}}_{-(-)^{k-k'}[\mathbf{d}K,L]^{\Delta}_{(1)}}\big)_{\boldsymbol{\theta}}(\sigma)\delta^{d_{\rm w}}(\sigma-\sigma') - (-)^{k-k'}\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\rm w}}(\sigma-\sigma')\big([K,L]^{\Delta}_{(1)}\big)_{\boldsymbol{\theta}}(\sigma) \tag{2.134}$$

Proof The proof is very similar to that one of proposition 3b (2.148) and is therefore omitted at this place. □

Conjugate superfields. With $\boldsymbol{\theta}^{\mu} = \boldsymbol{d}^{\rm w}\sigma^{\mu}$ we have introduced anticommuting coordinates and it would be nice to extend the anti-bracket of the fields x^m and $\boldsymbol{x}^+_m$ to a super-antibracket of conjugate superfields. Remember, in the previous subsection we had the superfields $\Phi^m = x^m + \boldsymbol{\theta}\boldsymbol{c}^m$ and its conjugate $\boldsymbol{S}_m$. There we had one $\boldsymbol{\theta}$ and two component fields. In general the number of component fields has to exceed the worldvolume dimension $d_{\rm w}$ (the number of $\boldsymbol{\theta}$'s) by one, s.th. we have to introduce a lot of new fields to realize conjugate superfields. But before, let us define the fermionic integration measure $\mu(\boldsymbol{\theta})$ via

$$\int \mu(\boldsymbol{\theta})f(\boldsymbol{\theta}) = \frac{\partial}{\partial\boldsymbol{\theta}^{d_{\rm w}}}\cdots\frac{\partial}{\partial\boldsymbol{\theta}^1}f(\boldsymbol{\theta}) = \frac{1}{d_{\rm w}!}\epsilon^{\mu_1\ldots\mu_{d_{\rm w}}}\frac{\partial}{\partial\boldsymbol{\theta}^{\mu_1}}\cdots\frac{\partial}{\partial\boldsymbol{\theta}^{\mu_{d_{\rm w}}}}f(\boldsymbol{\theta}) \tag{2.135}$$

The corresponding $d_{\rm w}$-dimensional δ-function is

$$\begin{aligned}
\delta^{d_{\rm w}}(\boldsymbol{\theta}'-\boldsymbol{\theta}) &\equiv (\boldsymbol{\theta}'^1-\boldsymbol{\theta}^1)\cdots(\boldsymbol{\theta}'^{d_{\rm w}}-\boldsymbol{\theta}^{d_{\rm w}}) \\
&= \frac{1}{d_{\rm w}!}\epsilon_{\mu_1\ldots\mu_{d_{\rm w}}}(\boldsymbol{\theta}'^{\mu_1}-\boldsymbol{\theta}^{\mu_1})\cdots(\boldsymbol{\theta}'^{\mu_{d_{\rm w}}}-\boldsymbol{\theta}^{\mu_{d_{\rm w}}}) \\
&= \sum_{k=0}^{d_{\rm w}}\frac{1}{k!(d_{\rm w}-k)!}\epsilon_{\mu_1\ldots\mu_{d_{\rm w}}}\boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_k}\boldsymbol{\theta}^{\mu_{k+1}}\cdots\boldsymbol{\theta}^{\mu_{d_{\rm w}}} \\
\int\mu(\boldsymbol{\theta}')\delta^{d_{\rm w}}(\boldsymbol{\theta}'-\boldsymbol{\theta})f(\boldsymbol{\theta}') &= f(\boldsymbol{\theta}) \\
\delta^{d_{\rm w}}(\boldsymbol{\theta}'-\boldsymbol{\theta}) &= (-)^{d_{\rm w}}\delta^{d_{\rm w}}(\boldsymbol{\theta}-\boldsymbol{\theta}')
\end{aligned} \tag{2.136}$$

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For the two conjugate superfields, call them Φ^m and $\boldsymbol{\Phi}^+_m$, we want to have the canonical super anti bracket

$$\left(\boldsymbol{\Phi}^+_m(\sigma', \boldsymbol{\theta}'),\Phi^n(\sigma,\boldsymbol{\theta})\right) = \delta^n_m \delta^{d_{\mathrm{w}}}(\sigma' - \sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}' - \boldsymbol{\theta}) = -\left(\Phi^n(\sigma,\boldsymbol{\theta}), \boldsymbol{\Phi}^+_m(\sigma',\boldsymbol{\theta}')\right) \tag{2.137}$$

From the above considerations about the fermionic delta function it is now clear, how these superfields can be defined (they are known as *de Rham superfields*, because of the interpretation of $\boldsymbol{\theta}^\mu$ as $\boldsymbol{d}^{\mathrm{w}}\sigma^\mu$; see e.g. [24, 19]):

$$\begin{aligned}
\Phi^m(\sigma,\boldsymbol{\theta}) \equiv\; & x^m(\sigma) + \boldsymbol{x}^m_{\mu_{d_{\mathrm{w}}}}(\sigma)\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}} + x^m_{\mu_{d_{\mathrm{w}}-1}\mu_{d_{\mathrm{w}}}}(\sigma)\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}-1}}\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}} + \ldots \\
& + x^m_{\mu_1\ldots\mu_{d_{\mathrm{w}}}}(\sigma)\boldsymbol{\theta}^{\mu_1}\cdots\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}}
\end{aligned} \tag{2.138}$$

$$\begin{aligned}
\boldsymbol{\Phi}^+_m(\sigma',\boldsymbol{\theta}') \equiv\; & \frac{1}{d_{\mathrm{w}}!}\epsilon_{\mu_1\ldots\mu_{d_{\mathrm{w}}}}\boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_{d_{\mathrm{w}}}}\boldsymbol{x}^+_m(\sigma') + \frac{1}{(d_{\mathrm{w}}-1)!1!}\epsilon_{\mu_1\ldots\mu_{d_{\mathrm{w}}}}\boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_{d_{\mathrm{w}}-1}}x^{+\,\mu_{d_{\mathrm{w}}}}_m(\sigma') \\
& + \frac{1}{(d_{\mathrm{w}}-2)!2!}\epsilon_{\mu_1\ldots\mu_{d_{\mathrm{w}}}}\boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_{d_{\mathrm{w}}-2}}\boldsymbol{x}^{+\,\mu_{d_{\mathrm{w}}-1}\mu_{d_{\mathrm{w}}}}_m(\sigma') + \ldots \\
& + \frac{1}{d_{\mathrm{w}}!}\epsilon_{\mu_1\ldots\mu_{d_{\mathrm{w}}}}\boldsymbol{x}^{+\,\mu_1\ldots\mu_{d_{\mathrm{w}}}}_m(\sigma')
\end{aligned} \tag{2.139}$$

The component fields with the matching number of worldsheet indices are conjugate to each other, e.g.

$$\left(\boldsymbol{x}^{+\,\mu_1\mu_2}_m(\sigma'),x^n_{\nu_1\nu_2}(\sigma)\right) = \delta^n_m\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\delta^{d_{\mathrm{w}}}(\sigma-\sigma') \tag{2.140}$$

For the notation with boldface symbols for anticommuting variables, the worldvolume was assumed to be even-dimensional. In this case, one can analytically continue the coordinate form of multivector-valued forms of the form

$$K^{(k,k')}(x,\mathbf{d}x,\boldsymbol{\partial}) \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\mathbf{d}x^{m_1}\wedge\cdots\wedge\mathbf{d}x^{m_k}\wedge\boldsymbol{\partial}_{n_1}\wedge\cdots\wedge\boldsymbol{\partial}_{n_{k'}} \tag{2.141}$$

to functions of superfields (in odd worldvolume dimension one would get a symmetrization of the multivector-indices) and redefine $K(\sigma,\boldsymbol{\theta})$ of (2.78) to

$$K^{(k,k')}(\sigma,\boldsymbol{\theta}) \equiv K^{(k,k')}\left(\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta})\right) \tag{2.142}$$

$$= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(\Phi)\boldsymbol{d}^{\mathrm{w}}\Phi^{m_1}\cdots\boldsymbol{d}^{\mathrm{w}}\Phi^{m_k}\boldsymbol{\Phi}^+_{n_1}\cdots\boldsymbol{\Phi}^+_{n_{k'}} \tag{2.143}$$

All other geometric quantities have to be understood in this new sense now:

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) \equiv \underbrace{T^{(t,t',t'')}\left(\Phi(\sigma,\boldsymbol{\theta}),\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right)}_{\overset{\boldsymbol{\theta}\equiv 0}{=}T^{(t,t',t'')}(\sigma)} \quad \text{(see (2.58))} \tag{2.144}$$

To stay with the examples used in (2.77)–(2.83):

$$\text{e.g.}\quad \mathbf{d}K(\sigma,\boldsymbol{\theta}) \equiv \mathbf{d}K\left(\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \quad \text{(compare (2.32))} \tag{2.145}$$

$$\begin{aligned}
\text{or } \boldsymbol{o}(\sigma,\boldsymbol{\theta}) &\equiv \boldsymbol{o}\left(\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \\
&= \boldsymbol{d}^{\mathrm{w}}\Phi^m(\sigma,\boldsymbol{\theta})\boldsymbol{d}^{\mathrm{w}}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) \quad (\text{compare } \boldsymbol{o}=\boldsymbol{c}^m p_m)
\end{aligned} \tag{2.146}$$

$$\begin{aligned}
\left[K^{(k,k')}{}_{,\mathbf{d}}\,L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) &\equiv \left[K^{(k,k')},L^{(l,l')}\right]^{(\Delta)}_{(1)}\left(\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \\
\mathbf{d}x^m(\sigma,\boldsymbol{\theta}) &\equiv \boldsymbol{d}^{\mathrm{w}}\Phi^m(\sigma,\boldsymbol{\theta}) \\
(\mathbf{d}\boldsymbol{\partial}_m)(\sigma,\boldsymbol{\theta}) &\equiv (\mathbf{d}\boldsymbol{b}_m)(\sigma,\boldsymbol{\theta}) \equiv \boldsymbol{d}^{\mathrm{w}}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta})
\end{aligned} \tag{2.147}$$

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Note that the former relation $K(\sigma,\boldsymbol{\theta}) = K(\sigma) + \boldsymbol{\theta}\mathbf{d}K(\sigma)$ does NOT hold any longer with those new definitions! Nevertheless we get a very similar statement as compared to propositions 2 on page 18:

Proposition 3b. For all multivector valued forms $K^{(k,k')}$, $L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$, the following equation holds for even worldvolume-dimension d_{w} for the corresponding superfields (2.142):

$$\begin{aligned}(K(\sigma',\boldsymbol{\theta'}),L(\sigma,\boldsymbol{\theta})) &= \delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta'}-\boldsymbol{\theta})\underbrace{[K,_{\mathbf{d}}L]^{\Delta}_{(1)}}_{-(-)^{k-k'}[\mathbf{d}K,L]^{\Delta}_{(1)}}(\sigma,\boldsymbol{\theta}) + \\ &\quad -(-)^{k-k'}\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma-\sigma')\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta'}-\boldsymbol{\theta})\,[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \qquad (2.148)\end{aligned}$$

where $[K,L]^{\Delta}_{(1)}$ is the big bracket (2.23) and $[K,_{\mathbf{d}}L]^{\Delta}_{(1)}$ is the derived bracket of the big bracket (2.49).

Note that σ and $\boldsymbol{\theta}$ have switched their roles compared to the previous subsection (2.89), where the algebraic bracket came together with the derivative with respect to $\boldsymbol{\theta}$ of the delta-functions, while now it comes along with ∂_μ of the delta-functions.

Proof Let us use again the second idea in the proof of proposition 2, i.e. first collect the terms with derivatives of the delta function, only to show that one gets the algebraic bracket, and after that argue that the term with plain delta functions is its derived bracket. In doing this, however, we will need to prove an extension of the above proposition to objects like $\mathbf{d}K$ (or more general an object $T^{(t,t',t'')}$ as in (2.28)) that contain the basis element p_m, which is then replaced by $\boldsymbol{d}^{\mathrm{w}}\boldsymbol{S}_m$ as e.g. in (2.145).
(i) The antibracket between two such objects T and $\tilde{T}$ gets contributions to the derivative of the delta-function only from the antibrackets between $\boldsymbol{d}^{\mathrm{w}}\Phi^m$ and $\boldsymbol{\Phi}^+_m$ and between Φ^m and $\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m$ (compare (2.137))

$$\begin{aligned}\left(\boldsymbol{\Phi}^+_m(\sigma',\boldsymbol{\theta'}),\boldsymbol{d}^{\mathrm{w}}\Phi^n(\sigma,\boldsymbol{\theta})\right) &= \delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta'}-\boldsymbol{\theta}) &(2.149)\\ \left(\boldsymbol{d}^{\mathrm{w}}\Phi^n(\sigma',\boldsymbol{\theta'}),\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta})\right) &= \delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta'}-\boldsymbol{\theta}) &(2.150)\\ \left(\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m(\sigma',\boldsymbol{\theta'}),\Phi^n(\sigma,\boldsymbol{\theta})\right) &= -\delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta'}-\boldsymbol{\theta}) &(2.151)\\ \left(\Phi^n(\sigma',\boldsymbol{\theta'}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta})\right) &= -\boldsymbol{\theta}^{\mu}\left(\Phi^n(\sigma',\boldsymbol{\theta'}),\partial_{\mu}\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta})\right)\\ &= \delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta'}-\boldsymbol{\theta}) &(2.152)\end{aligned}$$

The last case is the only one where we had to take care of an extra sign stemming from $\boldsymbol{\theta}$ jumping over the graded comma. Comparing this to (2.5), where we had

$$\begin{aligned}\{\boldsymbol{b}_m,\boldsymbol{c}^n\} &= \delta^n_m &(2.153)\\ \{\boldsymbol{c}^n,\boldsymbol{b}_m\} &= \delta^n_m &(2.154)\\ \{p_m,x^n\} &= \delta^n_m &(2.155)\\ \{x^n,p_m\} &= -\delta^n_m &(2.156)\end{aligned}$$

one recognizes that the only difference is an overall odd factor $\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta'}-\boldsymbol{\theta})$ (the delta-function for $\boldsymbol{\theta}$ is an even object for even worldvolume dimension d_{w}) and

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an additional minus sign for the lower two lines, but the corresponding indices just get contracted like for the Poisson bracket. After such a bracket of basis elements has been calculated (which happens just between the remaining factors of T (at σ') on the left and the remaining factors of $\tilde{T}$ (at σ) on the right) this overall odd factor has to be pulled to the very left which gives an additional factor of $(-)^{t-t'}$ (in the notation of (2.28)) plus an additional minus sign for the upper two lines which compensates the relative minus sign of before and we get just an overall factor of $-(-)^{t-t'}\boldsymbol{\theta}^\mu\partial_\mu\delta^{d_{\rm w}}(\sigma'-\sigma)\delta^{d_{\rm w}}(\boldsymbol{\theta}'-\boldsymbol{\theta})$ in all cases at the very left as compared to the Poisson-bracket. The remaining terms are still partly at σ and partly at σ', but using

$$A(\sigma)B(\sigma')\partial_\mu\delta(\sigma-\sigma') = A(\sigma)\partial_\mu B(\sigma)\delta(\sigma-\sigma') + A(\sigma)B(\sigma)\partial_\mu\delta(\sigma-\sigma') \quad \forall A,B \tag{2.157}$$

we can take all remaining factors in $T(\sigma',\boldsymbol{\theta}')$ at σ, while $\boldsymbol{\theta}'$ is set to $\boldsymbol{\theta}$ anyway by the δ-function. We have thus verified one of the coefficients of the complete antibracket:

$$\begin{aligned}(T(\sigma',\boldsymbol{\theta}'),\tilde{T}(\sigma,\boldsymbol{\theta})) &= -(-)^{t-t'}\boldsymbol{\theta}^\mu\partial_\mu\delta^{d_{\rm w}}(\sigma-\sigma')\delta^{d_{\rm w}}(\boldsymbol{\theta}'-\boldsymbol{\theta})\left[T,\tilde{T}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \\ &\quad +\delta^{d_{\rm w}}(\sigma-\sigma')\delta^{d_{\rm w}}(\boldsymbol{\theta}'-\boldsymbol{\theta})A(\sigma,\boldsymbol{\theta})\end{aligned} \tag{2.158}$$

with $A(\sigma,\boldsymbol{\theta})$ yet to be determined.
(ii) It remains to show that $A(\sigma,\boldsymbol{\theta})$ is a derived expression of $\left[T,\tilde{T}\right]^{\Delta}_{(1)}$. A hint to this fact is already given in (2.157), but this is not enough, as there is also a contribution from the $(\Phi^m,\boldsymbol{\Phi}^+_n)$-brackets. In order to get a precise relation between $A(\sigma,\boldsymbol{\theta})$ and $\left[T,\tilde{T}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})$, let us see how one can extract them from the complete antibracket. In order to hit the delta functions with the integration, it is enough to integrate over the patch $U(\sigma)$ containing the point which is parametrized by σ^μ. The last term in (2.158) is the only one contributing when integrating over σ' and $\boldsymbol{\theta}$

$$A(\sigma,\boldsymbol{\theta}) = \int_{U(\sigma)} \mathbf{d}^{d_{\rm w}}\sigma' \int \mu(\boldsymbol{\theta}') \quad (T(\sigma',\boldsymbol{\theta}'),\tilde{T}(\sigma,\boldsymbol{\theta})) \tag{2.159}$$

That the first term on the righthand side of (2.158) does not contribute is not obvious as $U(\sigma)$ might have a boundary. However, for this term one ends up integrating a $d_{\rm w}$-dimensional delta-function over a boundary of dimension not higher than $d_{\rm w}-1$, so that one is left with an at least one-dimensional delta-function on the boundary which vanishes as the boundary of the open neighbourhood $U(\sigma)$ of σ of course nowhere hits σ.

Extracting the algebraic bracket $\left[T,\tilde{T}\right]^{\Delta}_{(1)}$ is a bit more tricky. One can do it via

$$\begin{aligned}&\text{for any fixed index } \lambda : \left[T,\tilde{T}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) = \\ &\qquad -(-)^{t-t'}\int_{U(\sigma)}\mathbf{d}^{d_{\rm w}}\sigma'\int\mu(\boldsymbol{\theta}')\left(\frac{e^{\sigma'^\lambda}}{e^{\sigma^\lambda}}-1\right)\frac{\partial}{\partial\boldsymbol{\theta}^\lambda}(T(\sigma',\boldsymbol{\theta}'),\tilde{T}(\sigma,\boldsymbol{\theta}))\end{aligned} \tag{2.160}$$

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The boundary term proportional to $\left(\frac{e^{\sigma'\lambda}}{e^{\sigma\lambda}}-1\right)\delta^{d_{\rm w}}(\sigma-\sigma')$ appearing above on the right-hand side after partial integration vanishes as σ' in the prefactor is set to σ via the delta function.

The claim is now that $A(\sigma,\boldsymbol{\theta}) = -(-)^{t-t'}\left[\mathbf{d}T,\tilde{T}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})$. So let us calculate the righthand side via (2.160):

$$\begin{aligned}\left[\mathbf{d}T,\tilde{T}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) &= -(-)^{t+1-t'}\int_{U(\sigma)}\mathbf{d}^{d_{\rm w}}\sigma'\int\mu(\boldsymbol{\theta}')\left(\frac{e^{\sigma'\lambda}}{e^{\sigma\lambda}}-1\right)\frac{\partial}{\partial\boldsymbol{\theta}^{\lambda}}(\mathbf{d}T(\sigma',\boldsymbol{\theta}'),\tilde{T}(\sigma,\boldsymbol{\theta}))\\ &= -(-)^{t+1-t'}\int\mathbf{d}^{d_{\rm w}}\sigma'\int\mu(\boldsymbol{\theta}')\left(\frac{e^{\sigma'\lambda}}{e^{\sigma\lambda}}-1\right)\frac{\partial}{\partial\boldsymbol{\theta}^{\lambda}}\boldsymbol{\theta}'^{\mu}\partial'_{\mu}(T(\sigma',\boldsymbol{\theta}'),\tilde{T}(\sigma,\boldsymbol{\theta}))\end{aligned}$$

$(T,\tilde{T})$ contains in both terms a plain δ-function for the fermionic variables $\boldsymbol{\theta}$, so that we can replace $\boldsymbol{\theta}'$ by $\boldsymbol{\theta}$. Integration by parts of ∂'_{μ} (where possible boundary terms again do not contribute because of the vanishing of the delta function and its derivative on the boundary) delivers the desired result

$$\left[\mathbf{d}T,\tilde{T}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) = -(-)^{t-t'}\int\mathbf{d}^{d_{\rm w}}\sigma'\int\mu(\boldsymbol{\theta}')\quad(T(\sigma',\boldsymbol{\theta}'),\tilde{T}(\sigma,\boldsymbol{\theta})) = -(-)^{t-t'}A(\sigma,\boldsymbol{\theta}) \tag{2.161}$$

This completes the proof of proposition 3b. □

3. Applications in string theory or 2d CFT

In the previous section the dimension of the worldvolume was arbitrary or even dimensional. The appearance of derived brackets (including e.g. the Dorfman bracket) is thus not a special feature of a 2-dimensional sigma-model like string theory. There are, however, special features in string theory. Currents in string theory (which have conformal weight one) naturally are sums of 1-forms and vectors, if one takes the identification $\partial_1 x^m(\sigma)\leftrightarrow\mathbf{d}x^m$ and $p_m(\sigma)\leftrightarrow\boldsymbol{\partial}_m$, as in [2] (see footnote 14), e.g. $\partial x^m = \partial_1 x^m - \partial_0 x^m \hat{=} \mathbf{d}x^m - \eta^{mn}\boldsymbol{\partial}_n$. This is closely related to the identification in our previous section in the antifield formalism. In addition, only in two dimensions a single $\boldsymbol{\theta}$ can be interpreted as a worldsheet Weyl spinor (in 1 dimension it can be seen as a Dirac-spinor, but in higher dimensions the interpretation of $\boldsymbol{\theta}$ as worldvolume spinor breaks down). As we ended the last section with the antifield formalism, which therefore is perhaps still more present, let us start this section in the reversed order, beginning with the application in the antifield formalism.

3.1 Poisson sigma-model and Zucchini's "Hitchin sigma-model"

Remember for a moment the Poisson-σ-model [25, 24]. It is a two-dimensional sigma-model ($d_{\rm w}=2$) of the form

$$S_0 = \int_{\Sigma}\boldsymbol{\eta}_m\boldsymbol{d}^{\rm w}x^m + \frac{1}{2}P^{mn}(x)\boldsymbol{\eta}_m\boldsymbol{\eta}_n \tag{3.1}$$

where $\boldsymbol{\eta}_m$ is a worldsheet one-form. This model is topological if and only if the Poisson-structure $P^{mn}(x)$ is integrable, i.e. the Schouten-bracket of P with itself vanishes

$$S_0 \text{ topological} \iff [P,P]=0 \tag{3.2}$$

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It gives on the one hand a field theoretic implementation of Kontsevich's star product [24] and is on the other hand related to string theory via a topological limit (big antisymmetric part in the open string metric), which leads to the relation between string theory and noncommutative geometry.

The necessary ghost fields for the action can be introduced by extending x and η to de Rham superfields as in (2.138), (2.139)

$$\Phi^m(\sigma,\boldsymbol{\theta}) \equiv x^m(\sigma) + \underbrace{\boldsymbol{x}_\mu^m(\sigma)}_{\epsilon_{\mu\nu}\boldsymbol{\eta}^{+\nu n}}\boldsymbol{\theta}^\mu + \underbrace{x^m_{\mu_1\mu_2}(\sigma)}_{-\frac{1}{2}\varepsilon_{\mu_1\mu_2}\beta^{+\,m}}\ \boldsymbol{\theta}^{\mu_1}\boldsymbol{\theta}^{\mu_2} \tag{3.3}$$

$$\boldsymbol{\Phi}^+_m(\sigma',\boldsymbol{\theta}') \equiv \underbrace{\frac{1}{2!}\epsilon_{\mu_1\mu_2}\boldsymbol{x}_m^{+\,\mu_1\mu_2}(\sigma')}_{\equiv\boldsymbol{\beta}_m(\sigma')} + \boldsymbol{\theta}'^{\mu_1}\underbrace{\epsilon_{\mu_1\mu_2}x_m^{+\,\mu_2}(\sigma')}_{\eta_{\mu_1 m}} + \frac{1}{2}\epsilon_{\mu_1\mu_2}\boldsymbol{\theta}'^{\mu_1}\boldsymbol{\theta}'^{\mu_2}\boldsymbol{x}^+_m(\sigma') \tag{3.4}$$

One can use Hodge-duality to rename some component fields as indicated. $\boldsymbol{\beta}_m$ is then the ghost field related to the gauge symmetry. The action including ghost fields and antifields simply reads

$$S = \int d^2\sigma \int \mu(\boldsymbol{\theta}) \quad \boldsymbol{\Phi}^+_m \boldsymbol{d}^{\mathrm{w}}\Phi^m + \frac{1}{2}P^{mn}(\Phi)\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_n \tag{3.5}$$

The expression under the integral corresponds to the tensor $-\delta_m{}^n\mathbf{d}x^m\wedge\boldsymbol{\partial}_n + \frac{1}{2}P^{mn}\boldsymbol{\partial}_m\wedge\boldsymbol{\partial}_n$ and the antibracket in the master-equation (S,S) implements the Schoutenbracket on P, which is a well known relation. Therefore we will concentrate on a second example, which is very similar, but less known.

Zucchini suggested in [19] a 2-dimensional sigma-model which is topological if a generalized complex structure in the target space is integrable (see subsection C.2 on page 53 and C.4 on page 58 to learn more about generalized complex structures). His model is of the form

$$S = \int d^2\sigma \int \mu(\boldsymbol{\theta}) \ \left(\boldsymbol{\Phi}^+_m\boldsymbol{d}^{\mathrm{w}}\Phi^m + \right) \ \tfrac{1}{2}P^{mn}(\Phi)\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_n - \tfrac{1}{2}Q_{mn}(\Phi)\boldsymbol{d}^{\mathrm{w}}\Phi^m\boldsymbol{d}^{\mathrm{w}}\Phi^n - J^n{}_m\boldsymbol{d}^{\mathrm{w}}\Phi^m\boldsymbol{\Phi}^+_n \tag{3.6}$$

where P^{mn}, Q_{mn} and $J^m{}_n$ are the building blocks of the generalized complex structure (C.22)

$$\mathcal{J}^M{}_N = \begin{pmatrix} J^m{}_n & P^{mn} \\ -Q_{mn} & -J^n{}_m \end{pmatrix} \tag{3.7}$$

The first term of (3.6) can be absorbed by a field redefinition as already observed in [20]. Ignoring thus the first term and using our notations of before, S can be rewritten as

$$S = \int d^2\sigma \int \mu(\boldsymbol{\theta}) \quad \frac{1}{2}\mathcal{J}(\Phi,\boldsymbol{d}^{\mathrm{w}}\Phi,\boldsymbol{\Phi}^+) \tag{3.8}$$

Calculating the master equation explicitely and collecting the terms which combine to the lengthy tensors for the integrability condition (see (C.60)–(C.61)) is quite cumbersome, so

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we can enjoy using instead proposition 3b on page 25. For a worldsheet without boundary its integrated version reads

$$\left(\int d^{d_{\rm w}}\sigma' \int \mu(\boldsymbol{\theta}')K(\sigma',\boldsymbol{\theta}'), \int \mathbf{d}^{d_{\rm w}}\sigma \int \mu(\boldsymbol{\theta})L(\sigma,\boldsymbol{\theta})\right) = \int d^{d_{\rm w}}\sigma \int \mu(\boldsymbol{\theta})\,[K,_{\mathbf{d}}L]^{\Delta}_{(1)}\,(\sigma,\boldsymbol{\theta}) \tag{3.9}$$

which leads to the relation

$$(S,S) = 0 \qquad \Longleftrightarrow \int d^2\sigma \int \mu(\boldsymbol{\theta})\,[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)}\,(\sigma,\boldsymbol{\theta}) = 0 \tag{3.10}$$

The derived bracket of the big bracket of $\mathcal{J}$ with itself contains already the Nijenhuis tensor (see in the appendix in equation (C.71) and in the discussion around)

$$[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} \;=\; \mathcal{N}_{M_1M_2M_3}\mathbf{t}^{M_1}\mathbf{t}^{M_2}\mathbf{t}^{M_3} - 4\mathcal{J}^{JI}\mathcal{J}_{IM}\mathbf{t}^{M}p_J \tag{3.11}$$

$$\stackrel{\mathcal{J}^2=-1}{=} \mathcal{N}_{M_1M_2M_3}\mathbf{t}^{M_1}\mathbf{t}^{M_2}\mathbf{t}^{M_3} + 4\boldsymbol{o} \tag{3.12}$$

$$\mathbf{t}^M \;=\; (\mathbf{d}x^m, \boldsymbol{\partial}_m), \qquad p_J = (p_j, 0) \tag{3.13}$$

$$\boldsymbol{o}(\mathbf{d}x,p) \;=\; \mathbf{d}x^m p_m \tag{3.14}$$

For $\mathcal{J}^2 = -1$ the last term is proportional to the generator $\boldsymbol{o}$ (remember (2.8)). In (3.10), however, it appears with $\mathbf{d}x$ and p replaced by the superfields as in (2.146)

$$\boldsymbol{o}(\sigma,\boldsymbol{\theta}) = \boldsymbol{d}^{\rm w}\Phi^m(\sigma,\boldsymbol{\theta})\boldsymbol{d}^{\rm w}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = -\boldsymbol{d}^{\rm w}(\boldsymbol{d}^{\rm w}\Phi^m(\sigma,\boldsymbol{\theta})\boldsymbol{S}_m(\sigma,\boldsymbol{\theta})) \tag{3.15}$$

which is a total worldsheet derivative and therefore drops during the integration. We are left with the generalized Nijenhuis tensor as a function of superfields

$$\mathcal{N}(\sigma,\boldsymbol{\theta}) = \mathcal{N}_{M_1M_2M_3}(\Phi)\underline{\mathbf{t}}^{M_1}\underline{\mathbf{t}}^{M_2}\underline{\mathbf{t}}^{M_3} \tag{3.16}$$

$$\text{with} \qquad \underline{\mathbf{t}}^M \equiv (\boldsymbol{d}^{\rm w}\Phi^m, \boldsymbol{\Phi}^+_m) \tag{3.17}$$

Written in small indices

$$\mathcal{N}(\sigma,\boldsymbol{\theta}) = \mathcal{N}_{m_1m_2m_3}(\Phi)\underbrace{\boldsymbol{d}^{\rm w}\Phi^{m_1}\boldsymbol{d}^{\rm w}\Phi^{m_1}\boldsymbol{d}^{\rm w}\Phi^{m_1}}_{=0} + 3\mathcal{N}^n{}_{m_1m_2}(\Phi)\boldsymbol{\Phi}^+_n\boldsymbol{d}^{\rm w}\Phi^{m_1}\boldsymbol{d}^{\rm w}\Phi^{m_2}$$
$$+3\mathcal{N}_n{}^{m_1m_2}(\Phi)\boldsymbol{d}^{\rm w}\Phi^n\boldsymbol{\Phi}^+_{m_1}\boldsymbol{\Phi}^+_{m_2} + \mathcal{N}^{m_1m_2m_3}(\Phi)\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_m \tag{3.18}$$

One realizes that the first term vanishes identically (as mentioned in [19]) and only the remaining three tensors are required to vanish in order to satisfy (3.10).

3.2 Relation between a second worldsheet supercharge and generalized complex geometry

In [15] the relation between an extended worldsheet supersymmetry in string theory and the presence of an integrable generalized complex structure was explored. Zabzine clarified in [18] the relation in an model independent way in a Hamiltonian description. The structures appearing there are almost the same that we have discussed before although we have to modify the procedure a little bit due to the interpretation of $\boldsymbol{\theta}$ as a worldsheet spinor.

Consider a sigma-model with 2-dimensional worldvolume (worldsheet) with manifest $N = 1$ supersymmetry on the worldsheet. In the phase space there is only one σ-coordinate left. Let us denote the corresponding superfields, following loosely [18], by

$$\Phi^m(\sigma, \boldsymbol{\theta}) \equiv x^m(\sigma) + \boldsymbol{\theta}\boldsymbol{\lambda}^m(\sigma) \tag{3.19}$$

$$\boldsymbol{S}_m(\sigma, \boldsymbol{\theta}) \equiv \boldsymbol{\rho}_m(\sigma) + \boldsymbol{\theta} p_m(\sigma) \tag{3.20}$$

In comparison to section 2.3, there is a change of notation from $\boldsymbol{c}^m \to \boldsymbol{\lambda}^m$ and $\boldsymbol{b}_m \to \boldsymbol{\rho}_m$ as $\boldsymbol{b}$ and $\boldsymbol{c}$ suggest the interpretation as ghosts which is not true in this case, where $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ are worldsheet fermions. Introduce now, following Zabzine, the generator $\mathrm{Q}_{\boldsymbol{\theta}}$ of the *manifest SUSY* and the corresponding *covariant derivative* $\mathrm{D}_{\boldsymbol{\theta}}$

$$\mathrm{Q}_{\boldsymbol{\theta}} \equiv \partial_{\boldsymbol{\theta}} + \boldsymbol{\theta}\partial_\sigma \qquad \mathrm{D}_{\boldsymbol{\theta}} \equiv \partial_{\boldsymbol{\theta}} - \boldsymbol{\theta}\partial_\sigma \tag{3.21}$$

with the SUSY algebra

$$[\mathrm{Q}_{\boldsymbol{\theta}}, \mathrm{Q}_{\boldsymbol{\theta}}] = 2\partial_\sigma = -[\mathrm{D}_{\boldsymbol{\theta}}, \mathrm{D}_{\boldsymbol{\theta}}] \qquad [\mathrm{Q}_{\boldsymbol{\theta}}, \mathrm{D}_{\boldsymbol{\theta}}] = 0 \tag{3.22}$$

$\mathrm{Q}_{\boldsymbol{\theta}}$ is the sum of two nilpotent differential operators, namely $\partial_{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}\partial_\sigma$. Acting on the Superfields Φ^m and $\boldsymbol{S}^m$, they induce the differentials $\mathbf{s}$ and $\tilde{\mathbf{s}}$ on the component fields, which are in turn generated via the Poisson bracket by phase space functions $\boldsymbol{\Omega}$ (the same as (2.66)) and $\tilde{\boldsymbol{\Omega}}$.

$$\boldsymbol{\Omega} \equiv \int d\sigma\, \boldsymbol{\lambda}^k p_k \tag{3.23}$$

$$\tilde{\boldsymbol{\Omega}} = -\int d\sigma\, \partial_\sigma x^k \boldsymbol{\rho}_k \tag{3.24}$$

$$\mathbf{s}x^m \equiv \{\boldsymbol{\Omega}, x^m\} = \boldsymbol{\lambda}^m \leftrightarrow \mathbf{d}x^m, \qquad \mathbf{s}\boldsymbol{\rho}_m \equiv \{\boldsymbol{\Omega}, \boldsymbol{\rho}_m\} = p_m \leftrightarrow \mathbf{d}(\boldsymbol{\partial}_m), \tag{3.25}$$

$$\tilde{\mathbf{s}}\boldsymbol{\lambda}^m \equiv \left\{\tilde{\boldsymbol{\Omega}}, \boldsymbol{\lambda}^m\right\} = -\partial_\sigma x^m, \qquad \tilde{\mathbf{s}}p_k = -\partial_\sigma \boldsymbol{\rho}_k = \left\{\tilde{\boldsymbol{\Omega}}, p_k\right\}, \tag{3.26}$$

$$\mathbf{s}\Phi^m = \partial_{\boldsymbol{\theta}}\Phi^m, \qquad \mathbf{s}\boldsymbol{S}_m = \partial_{\boldsymbol{\theta}}\boldsymbol{S}_m \qquad \tilde{\mathbf{s}}\Phi^m = \boldsymbol{\theta}\partial_\sigma\Phi^m, \qquad \tilde{\mathbf{s}}\boldsymbol{S}_m = \boldsymbol{\theta}\partial_\sigma\boldsymbol{S}_m \tag{3.27}$$

The Poisson-generator for the SUSY transformations of the component fields induced by[16] $\mathrm{Q}_{\boldsymbol{\theta}}$ is thus the sum of the generators of $\mathbf{s}$ and $\tilde{\mathbf{s}}$

$$\boldsymbol{Q} = \boldsymbol{\Omega} + \tilde{\boldsymbol{\Omega}} = \int d\sigma\, \boldsymbol{\lambda}^k p_k - \partial_\sigma x^k \boldsymbol{\rho}_k = -\int d\sigma \int d\boldsymbol{\theta}\, \mathrm{Q}_{\boldsymbol{\theta}}\Phi^k \boldsymbol{S}_k \tag{3.28}$$

In (2.73) superfields were defined via $\partial_{\boldsymbol{\theta}} Y = \mathbf{s}Y$ in order to implement the exterior derivative directly with $\partial_{\boldsymbol{\theta}}$. In that sense Φ, $\boldsymbol{S}$, $\mathbf{d}\Phi$, $\mathbf{d}\boldsymbol{S}$ and all analytic functions of them were

[16] We have

$$\begin{aligned}
\mathrm{Q}_{\boldsymbol{\theta}}\Phi^m &= \boldsymbol{\lambda}^m + \boldsymbol{\theta}\partial_\sigma x^m, \qquad \mathrm{Q}_{\boldsymbol{\theta}}\boldsymbol{S}_m = p_m + \boldsymbol{\theta}\partial_\sigma\boldsymbol{\rho}_m \\
\mathrm{D}_{\boldsymbol{\theta}}\Phi^m &= \boldsymbol{\lambda}^m(\sigma) - \boldsymbol{\theta}\partial_\sigma x^m, \qquad \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}_m = p_m - \boldsymbol{\theta}\partial_\sigma\boldsymbol{\rho}_m \\
\delta_{\boldsymbol{\varepsilon}} x^m &= \boldsymbol{\varepsilon}\boldsymbol{\lambda}^m, \qquad \delta_{\boldsymbol{\varepsilon}}\boldsymbol{\lambda}^m = -\boldsymbol{\varepsilon}\partial_\sigma x^m \\
\delta_{\boldsymbol{\varepsilon}}\boldsymbol{\rho}_m &= \boldsymbol{\varepsilon} p_m, \qquad \delta_{\boldsymbol{\varepsilon}} p_m = -\boldsymbol{\varepsilon}\partial_\sigma\boldsymbol{\rho}_m
\end{aligned}$$

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superfields. In the context of worldsheet supersymmetry, one prefers of course a supersymmetric covariant formulation. Let us therefore define in this subsection proper *superfields* via

$$Y \text{ is a superfiled} \quad :\Longleftrightarrow \quad \mathrm{Q}_{\boldsymbol{\theta}} Y \stackrel{!}{=} \{\boldsymbol{Q}, Y\} = (\mathbf{s} + \tilde{\mathbf{s}})Y \tag{3.29}$$

which holds for Φ, $\boldsymbol{S}$,$\mathrm{D}_{\boldsymbol{\theta}}\Phi$, $\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}$, all analytic functions of them (like our analytically continued multivector valued forms) and worldsheet spatial derivatives ∂_σ thereof (but not for e.g. $\mathrm{Q}_{\boldsymbol{\theta}}\Phi$. This means that although we have $\mathrm{Q}_{\boldsymbol{\theta}}\Phi = (\mathbf{s} + \tilde{\mathbf{s}})\Phi$ this does not hold for a second action, i.e. $\mathrm{Q}^2_{\boldsymbol{\theta}}\Phi \neq (\mathbf{s} + \tilde{\mathbf{s}})^2\Phi$, which explains the somewhat confusing fact that the Poisson-generator $\boldsymbol{Q}$ has the opposite sign in the algebra than $\mathrm{Q}_{\boldsymbol{\theta}}$

$$\{\boldsymbol{Q}, \boldsymbol{Q}\} = -2P \tag{3.30}$$

where we introduced the phase-space generator P for the worldsheet translation induced by ∂_σ

$$P \equiv \int d\sigma \quad \partial_\sigma x^k p_k + \partial_\sigma \boldsymbol{\lambda}^k \boldsymbol{\rho}_k = \int d\sigma \int d\boldsymbol{\theta} \quad \partial_\sigma \Phi^k \boldsymbol{S}_k \tag{3.31}$$

The same phenomenon appears for the differentials $\mathbf{s}$ and $\tilde{\mathbf{s}}$. The graded commutator of $\partial_{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}\partial_\sigma$ is the worldsheet derivative $[\partial_{\boldsymbol{\theta}}, \boldsymbol{\theta}\partial_\sigma] = \partial_\sigma$, while the algebra for $\mathbf{s}$ and $\tilde{\mathbf{s}}$ has the opposite sign

$$[\mathbf{s}, \tilde{\mathbf{s}}]\, Y(\sigma, \boldsymbol{\theta}) = -\partial_\sigma Y(\sigma, \boldsymbol{\theta}) \tag{3.32}$$

$$\mathbf{s}\tilde{\boldsymbol{\Omega}} = \left\{\boldsymbol{\Omega}, \tilde{\boldsymbol{\Omega}}\right\} = -P = \tilde{\mathbf{s}}\boldsymbol{\Omega} \tag{3.33}$$

One major statement in [18] is as follows: Making a general ansatz for a generator of a second, non-manifest supersymmetry, of the form (some signs are adopted to our conventions)

$$\boldsymbol{Q}_2 \equiv \frac{1}{2}\int d\sigma \int d\boldsymbol{\theta} \; (P^{mn}(\Phi)\boldsymbol{S}_m\boldsymbol{S}_n - Q_{mn}(\Phi)\mathrm{D}_{\boldsymbol{\theta}}\Phi^m\mathrm{D}_{\boldsymbol{\theta}}\Phi^n + 2J^m{}_n(\Phi)\boldsymbol{S}_m\mathrm{D}_{\boldsymbol{\theta}}\Phi^n) \tag{3.34}$$

and requiring the same algebra as for $\boldsymbol{Q}$ in (3.30)

$$\{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} = -2P \tag{3.35}$$

$$\Big(\{\boldsymbol{Q}, \boldsymbol{Q}_2\} = 0\Big) \tag{3.36}$$

is equivalent to

$$\mathcal{J}^M{}_N \equiv \begin{pmatrix} J^m{}_n & P^{mn} \\ -Q_{mn} & -J^n{}_m \end{pmatrix} \tag{3.37}$$

being an integrable generalized complex structure (see in the appendix C.2 on page 53 and C.4 on page 58). On a worldsheet without boundary, the second condition is actually superfluous, because it is already implemented via the ansatz: The expression in the integral

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is an analytic function of superfields and therefore a superfield itself. According to (3.29) we can replace at this point the commutator with $\boldsymbol{Q}$ with the action of $\mathrm{Q}_{\boldsymbol{\theta}}$ and get

$$\{\boldsymbol{Q},\boldsymbol{Q}_2\} = \int d\sigma \int d\boldsymbol{\theta} \quad \mathrm{Q}_{\boldsymbol{\theta}}(\ldots) = \int d\sigma \quad \partial_\sigma(\ldots) = 0 \tag{3.38}$$

For the other condition, the actual supersymmetry algebra (3.35), the aim of the present considerations should now be clear. The generalized complex structure $\mathcal{J}$ itself is a sum of multivector valued forms

$$\mathcal{J} \equiv \mathcal{J}^{MN}(x)\mathbf{t}_M\mathbf{t}_N \equiv P^{mn}(x)\boldsymbol{\partial}_m \wedge \boldsymbol{\partial}_n - Q_{mn}(x)\mathbf{d}x^m\mathbf{d}x^n + 2J^m{}_n(x)\boldsymbol{\partial}_m \wedge \mathbf{d}x^n \tag{3.39}$$

which can be seen as a function of x and the basis elements

$$\mathcal{J} = \mathcal{J}(x,\mathbf{d}x,\boldsymbol{\partial}) \tag{3.40}$$

In 2.3 we replaced the arguments of functions like this with "superfields" $x^m \to \Phi^m$, $\mathbf{d}x^m \to \partial_{\boldsymbol{\theta}}\Phi^m$ and $\boldsymbol{\partial}_m \to \boldsymbol{S}_m$. The name superfield might have been misleading, as $\partial_{\boldsymbol{\theta}}\Phi$ is only a superfield in the sense that it implements the target-space exterior derivative via $\partial_{\boldsymbol{\theta}}$, but it is not a superfield in the sense of worldsheet supersymmetry. In a supersymmetric theory one prefers a supersymmetric covariant formulation. Working with $\partial_{\boldsymbol{\theta}}\Phi$ as before is therefore not desirable and we replace $\partial_{\boldsymbol{\theta}}\Phi$ by $\mathrm{D}_{\boldsymbol{\theta}}\Phi$, leading directly to $\boldsymbol{Q}_2$ (3.34) which now can be written as

$$\boldsymbol{Q}_2 = \frac{1}{2}\int d\sigma \int d\boldsymbol{\theta}\, \mathcal{J}\left(\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \tag{3.41}$$

Apart from the change $\partial_{\boldsymbol{\theta}}\Phi \to \mathrm{D}_{\boldsymbol{\theta}}\Phi$ we expect from the previous section that the Poisson bracket of $\boldsymbol{Q}_2$ with itself induces some algebraic and some derived bracket of $\mathcal{J}$ with itself which then corresponds to the integrability condition for $\mathcal{J}$. This is indeed the case, but we first have to study the changes coming from $\partial_{\boldsymbol{\theta}}\Phi \to \mathrm{D}_{\boldsymbol{\theta}}\Phi$. In other words, we need a new formulation of proposition 1 (2.89) in the case of two-dimensional supersymmetry (Proposition 1 is of course still valid, but it is not formulated in a supersymmetric covariant way. It should, however, be applicable to e.g. BRST symmetries). Let us redefine the meaning of $K(\sigma,\boldsymbol{\theta})$ in (2.78) for a multivector valued form $K^{(k,k')}$

$$\begin{aligned} K^{(k,k')}(\sigma,\boldsymbol{\theta}) &\equiv K^{(k,k')}\left(\Phi^m(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta}), \boldsymbol{S}_m(\sigma,\boldsymbol{\theta})\right) \\ &= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(\Phi(\sigma,\boldsymbol{\theta})\right) \mathrm{D}_{\boldsymbol{\theta}}\Phi^{m_1}(\sigma,\boldsymbol{\theta})\ldots\mathrm{D}_{\boldsymbol{\theta}}\Phi^{m_k}(\sigma,\boldsymbol{\theta}) \times \\ &\quad \times\boldsymbol{S}_{n_1}(\sigma,\boldsymbol{\theta})\ldots\boldsymbol{S}_{n_{k'}}(\sigma,\boldsymbol{\theta}) \\ &\underset{(2.57)}{\overset{\boldsymbol{\theta}=0}{\to}} K^{(k,k')}(\sigma) \end{aligned} \tag{3.42}$$

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Likewise for all the other examples in (2.77)–(2.83):

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) \equiv \underbrace{T^{(t,t',t'')}\left(\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right)}_{\overset{\boldsymbol{\theta}=0}{\equiv} T^{(t,t',t'')}(\sigma)} \quad \text{(see (2.58))}$$

$$\text{e.g.} \quad \mathbf{d}K(\sigma,\boldsymbol{\theta}) \equiv \mathbf{d}K\left(\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right)$$

$$\text{or} \quad \boldsymbol{o}(\sigma,\boldsymbol{\theta}) \equiv \boldsymbol{o}\left(\mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \overset{(2.8)}{=} \mathrm{D}_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta})\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) \underset{(2.60)}{\overset{\boldsymbol{\theta}=0}{\equiv}} \boldsymbol{o}(\sigma)$$

$$\left[K^{(k,k')}\overset{\mathbf{,}}{_{\mathbf{d}}}\, L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \equiv \underbrace{\left[K^{(k,k')}\mathbf{,}L^{(l,l')}\right]^{(\Delta)}_{(1)}\left(\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right)}_{\underset{(2.61)}{\overset{\boldsymbol{\theta}=0}{\equiv}}\left[K^{(k,k')}\mathbf{,}L^{(l,l')}\right]^{(\Delta)}_{(1)}(\sigma)}$$

$$\mathbf{d}x^m(\sigma,\boldsymbol{\theta}) \equiv \mathrm{D}_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta}) = \boldsymbol{\lambda}^m(\sigma) - \boldsymbol{\theta}\partial_\sigma x^m(\sigma)$$
$$\mathbf{d}\boldsymbol{\partial}_m(\sigma,\boldsymbol{\theta}) \equiv \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = p_m(\sigma) - \boldsymbol{\theta}\partial_\sigma\boldsymbol{\rho}_m(\sigma)$$

Expanding K in $\boldsymbol{\theta}$ yields

$$K^{(k,k')}(\sigma,\boldsymbol{\theta}) = K^{(k,k')}(\sigma) + \boldsymbol{\theta}\left(\partial_{\boldsymbol{\theta}'}K^{(k,k')}(\sigma,\boldsymbol{\theta}')\Big|_{\boldsymbol{\theta}'=0}\right) \tag{3.43}$$
$$= K^{(k,k')}(\sigma) + \boldsymbol{\theta}\left(\mathrm{Q}_{\boldsymbol{\theta}'}K^{(k,k')}(\sigma,\boldsymbol{\theta}')\Big|_{\boldsymbol{\theta}'=0}\right) \tag{3.44}$$

As K is a superfield, we can replace $\mathrm{Q}_{\boldsymbol{\theta}}$ by $\mathbf{s}+\tilde{\mathbf{s}}$

$$K^{(k,k')}(\sigma,\boldsymbol{\theta}) = K^{(k,k')}(\sigma) + \boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})K^{(k,k')}(\sigma) \tag{3.45}$$
$$= K^{(k,k')}(\sigma) + \boldsymbol{\theta}\left((\mathbf{d}+\imath_v)K^{(k,k')}\right)(\sigma)\Big|_{v^k\to-\partial_\sigma x^k} \tag{3.46}$$

This is the analogue to the non-supersymmetric (2.87) and delivers the exterior derivative which will lead to the appearance of the derived bracket. The relation between $\tilde{\mathbf{s}}$ and the inner product with a vector should perhaps be clarified. Remember that all multivector forms at $\boldsymbol{\theta}=0$, $K^{(k,k')}(\sigma)$, are analytic functions of the component fields $x^m, \boldsymbol{\lambda}^m$ and $\boldsymbol{\rho}_m$. But among those fields, $\tilde{\mathbf{s}}$ acts only on $\boldsymbol{\lambda}^m$ and we can express it with partial derivatives (instead of functional ones) when acting on K:

$$\tilde{\mathbf{s}}K(\sigma) = -\partial_\sigma x^m \frac{\partial}{\partial\boldsymbol{\lambda}^m}K(x,\boldsymbol{\lambda},\boldsymbol{\rho}) = \imath_v K(\sigma)|_{v^k=-\partial_\sigma x^k} \tag{3.47}$$

in the Poisson bracket of $\tilde{\mathbf{s}}K$ with another multivector valued form L at $\boldsymbol{\theta}=0$, nothing acts on $v^k=-\partial_\sigma x^k$ (which would produce a derivative of a delta function), as L does not contain p_k. Therefore we have

$$\left\{\tilde{\mathbf{s}}K(\sigma'), L(\sigma)\right\} = [\imath_v K, L](\sigma)|_{v^k=-\partial_\sigma x^k}\,\delta(\sigma-\sigma') \tag{3.48}$$

which we will need below. For superfields we have $Y(\sigma,\boldsymbol{\theta}) = Y(\sigma)+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})Y(\sigma)$. Applying the same to v yields

$$v^k(\sigma)+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})v^k(\sigma) = -\partial_\sigma x^k - \boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\partial_\sigma x^k(\sigma) = -\partial_\sigma x^k - \boldsymbol{\theta}\partial_\sigma\lambda^k(\sigma) = -\partial_\sigma\Phi^k \tag{3.49}$$

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Proposition 1b. For all multivector valued forms $K^{(k,k')}$, $L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$, the following equation holds for the corresponding worldsheet-superfields (3.42)

$$\{K^{(k,k')}(\sigma',\boldsymbol{\theta}'),L^{(l,l')}(\sigma,\boldsymbol{\theta})\} = \mathrm{D}_{\boldsymbol{\theta}}\left(\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\sigma-\sigma')\right)[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})+\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\times \\ \times\delta(\sigma-\sigma')\Big(\underbrace{[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})}_{-(-)^{k-k'}[K,_{\mathbf{d}}L]^{\Delta}_{(1)}}+\underbrace{[\imath_v K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})}_{-(-)^{k-k'}[K,_{\imath_v}L]}\Big|_{v^k=-\partial_\sigma\Phi^k}\Big) \tag{3.50}$$

where e.g. $[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \equiv [\mathbf{d}K,L]^{\Delta}_{(1)}\left(\Phi(\sigma,\boldsymbol{\theta}),\mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right)$. The integrated version for a worldsheet without boundary reads

$$\Big\{\int d\sigma'\int d\boldsymbol{\theta}' K^{(k,k')}(\sigma',\boldsymbol{\theta}'),\int d\sigma\int d\boldsymbol{\theta}\, L^{(l,l')}(\sigma,\boldsymbol{\theta})\Big\} = \\ = (\mathbf{s}+\tilde{\mathbf{s}})\int d\sigma\left([K,_{\mathbf{d}}L]^{\Delta}_{(1)}-(-)^{k-k'}[\imath_v K,L]^{\Delta}_{(1)}\Big|_{v^k=-\partial_\sigma x^k}\right)(\sigma) \tag{3.51}$$

Proof Let us use (3.45) for both multivector valued fields and plug into the lefthand side of (3.50)

$$\begin{aligned}
&\big\{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\big\} = \\
&\quad= \big\{K(\sigma')+\boldsymbol{\theta}'(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma')\,,\,L(\sigma)+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})L(\sigma)\big\}\\
&\quad= \big\{K(\sigma'),L(\sigma)\big\}+\boldsymbol{\theta}'\big\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\big\}+(-)^{k-k'}\boldsymbol{\theta}\big\{K(\sigma'),\ (\mathbf{s}+\tilde{\mathbf{s}})L(\sigma)\big\}\\
&\qquad+(-)^{k-k'}\boldsymbol{\theta}\boldsymbol{\theta}'\big\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),(\mathbf{s}+\tilde{\mathbf{s}})L(\sigma)\big\}\\
&\quad= \big\{K(\sigma'),L(\sigma)\big\}+(\boldsymbol{\theta}'-\boldsymbol{\theta})\big\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\big\}+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\big\{K(\sigma'),L(\sigma)\big\}\\
&\qquad+\boldsymbol{\theta}'\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\big\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\big\}-\boldsymbol{\theta}'\boldsymbol{\theta}\big\{(\mathbf{s}+\tilde{\mathbf{s}})(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\big\}\\
&\quad= (1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))\big\{K(\sigma'),L(\sigma)\big\}+(\boldsymbol{\theta}'-\boldsymbol{\theta})\,(1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))\big\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\big\}\\
&\qquad-\boldsymbol{\theta}'\boldsymbol{\theta}\big\{\underbrace{[\mathbf{s},\tilde{\mathbf{s}}]}_{-\partial_{\sigma'}}K(\sigma'),L(\sigma)\big\}\\
&\quad= \delta(\sigma-\sigma')\,(1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))\,[K,L]^{\Delta}_{(1)}(\sigma)+(\boldsymbol{\theta}'-\boldsymbol{\theta})\,(1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))\big\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\big\}\\
&\qquad-(\boldsymbol{\theta}'-\boldsymbol{\theta})\boldsymbol{\theta}\partial_\sigma\delta(\sigma-\sigma')\,[K,L]^{\Delta}_{(1)}(\sigma)
\end{aligned}$$

Now let us make use of (3.48) and (3.49) to arrive at

$$\big\{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\big\} = \mathrm{D}_{\boldsymbol{\theta}}\left(\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\sigma-\sigma')\right)[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \\ +\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')\left[(\mathbf{d}+\imath_v)K,L\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})\Big|_{v^k=-\partial_\sigma\Phi^k}$$

which is the first equation of the proposition. Integrating over $\boldsymbol{\theta}'$ and σ' results in

$$\begin{aligned}
\int d\sigma'\int d\boldsymbol{\theta}'\big\{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\big\} &= \left[(\mathbf{d}+\imath_v)K,L\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})\Big|_{v^k=-\partial_\sigma\Phi^k}\\
&= \left[(\mathbf{d}+\imath_v)K,L\right]^{\Delta}_{(1)}(\sigma)\Big|_{v^k=-\partial_\sigma x^k}\\
&\quad+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\left[(\mathbf{d}+\imath_v)K,L\right]^{\Delta}_{(1)}(\sigma)\Big|_{v^k=-\partial_\sigma x^k}
\end{aligned}$$

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A second integration picks out the linear part in $\boldsymbol{\theta}$ and adjusting the order of the integrations gives the additional sign in (3.51). □

Application to the second supercharge $\boldsymbol{Q}_2$. We are now ready to apply the proposition in the integrated form (3.51) to the question of the existence of a second worldsheet supersymmetry $\boldsymbol{Q}_2$. Remember, we want $\{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} = -2P$. Due to the proposition, the lefthand side can be written as

$$\{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} = \frac{1}{4}(\mathbf{s} + \tilde{\mathbf{s}}) \int d\sigma \left([\mathcal{J}, _{\mathbf{d}} \mathcal{J}]^{\Delta}_{(1)} - [\imath_v \mathcal{J}, \mathcal{J}]^{\Delta}_{(1)} \Big|_{v=-\partial_\sigma x^k \boldsymbol{\rho}_k} \right)(\sigma) \tag{3.52}$$

For $\mathcal{J}^2 = -1$, the second term under the integral simplifies significantly

$$\begin{aligned} -\frac{1}{4} \int d\sigma [\imath_v \mathcal{J}, \mathcal{J}]^{\Delta}_{(1)} \Big|_{v=-\partial_\sigma x^k \boldsymbol{\rho}_k} &= -\int d\sigma \, v^K \mathcal{J}_K{}^L \mathcal{J}_L{}^M \mathbf{t}_M \Big|_{v=-\partial_\sigma x^k \boldsymbol{\rho}_k} \\ &= -\int d\sigma \, \partial_\sigma x^k \boldsymbol{\rho}_k = \tilde{\boldsymbol{\Omega}} \end{aligned} \tag{3.53}$$

Recalling that

$$(\mathbf{s} + \tilde{\mathbf{s}})\tilde{\boldsymbol{\Omega}} = \mathbf{s}\tilde{\boldsymbol{\Omega}} = \tilde{\mathbf{s}}\boldsymbol{\Omega} = (\mathbf{s} + \tilde{\mathbf{s}})\boldsymbol{\Omega} = -P \tag{3.54}$$

$$\text{and} \qquad \boldsymbol{\Omega} = \int d\sigma \, \boldsymbol{o}(\sigma) \quad (\text{see (2.60)}) \tag{3.55}$$

we can rewrite (3.52) as

$$\Rightarrow \{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} = \frac{1}{4}(\mathbf{s} + \tilde{\mathbf{s}}) \left(\int d\sigma \, [\mathcal{J}, _{\mathbf{d}} \mathcal{J}]^{\Delta}_{(1)} + 4\boldsymbol{\Omega} \right) \tag{3.56}$$

$$= \frac{1}{4}(\mathbf{s} + \tilde{\mathbf{s}}) \left(\int d\sigma \, \left([\mathcal{J}, _{\mathbf{d}} \mathcal{J}]^{\Delta}_{(1)} - 4\boldsymbol{o} \right)(\sigma) \right) + 2 \underbrace{\tilde{\mathbf{s}}\boldsymbol{\Omega}}_{-P} \tag{3.57}$$

The righthand side clearly equals $-2P$ for

$$[\mathcal{J}, _{\mathbf{d}} \mathcal{J}]^{\Delta}_{(1)} - 4\boldsymbol{o} = 0 \tag{3.58}$$

which is again (according to (C.96)) just the integrability condition for the generalized almost complex structure $\mathcal{J}$.

4. Conclusions

We have seen two closely related mechanisms in sigma-models with a special field content which lead to the derived bracket of the target space algebraic bracket by the target space exterior derivative. This exterior derivative is implemented in the sigma model in one case via the derivative with respect to a (worldvolume-) Grassmann coordinate and in the other case via the derivative with respect to the worldvolume coordinate itself. In the latter case this derivative has to be contracted with (worldvolume-) Grassmann coordinates in order to be an odd differential. This leads to the problem that higher powers of the basis elements vanish, as soon as the power exceeds the worldvolume dimension as it happens

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in Zucchini's application. A big number of Grassmann-variables is therefore advantageous in that approach. For the other mechanism one rather prefers to have only one single Grassmann variable as there is no need for any contraction. There is one worldvolume dimension more in the Lagrangian formalism and for that reason it was preferable to apply there the mechanism with worldvolume derivatives and use the other one in the Hamiltonian formalism.

If one does not consider antisymmetric tensors of higher rank, but only vectors or one-forms (or forms of worldvolume-dimension), the partial worldvolume derivative without a Grassmann-coordinate is enough. There is either no need for antisymmetrization or it can be performed with the worldvolume epsilon tensor. The nature of the mechanism remains the same and leads to the observations in [2, 4] that the Poisson bracket implements the Dorfman bracket for sums of vectors and one-forms and the corresponding derived bracket for sums of vectors and p-forms on a p-brane [4]. In that sense, the present article is a generalization of those observations.

There remain a couple of things to do. It should be possible to implement in the same manner by e.g. a BRST differential other target space differentials which can depend on some extra-structure and repeat the same analysis. Symmetric tensors then become more interesting as well, because they need such an extra-structure anyway for a meaningful differential. From the string theory point of view, the application of extended worldsheet supersymmetry corresponds to applications in the RNS string. But generalized complex geometry contains the tools to allow RR-fluxes, which are hard to treat in RNS. It would therefore be nice to find some topological limit in a string theory formalism which is extendable to RR-fields, like the Berkovits-string [26], leading to a topological sigma model like Zucchini's, in order to learn more about the correspondence between string theory and generalized complex geometry.

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A. Notation and conventions

Within the article, a lot of different types of tensors have to be denoted. The choices and sometimes some logic behind, will be presented here.

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World-volume-coordinates are denoted by σ^μ, target-space coordinates by x^m, target space vector-fields by $a, b, \ldots$ or $v, w, \ldots$, 1-forms by small Greek letters $\alpha, \beta, \ldots$ and generalized $T \oplus T^*$-vectors by $\mathfrak{a}, \mathfrak{b}, \ldots$ or $\mathfrak{v}, \mathfrak{w}, \ldots$. For an explicit split in vector and 1-form, the letters from the beginning of the alphabet are better suited, as there is a better correspondence between Latin and Greek symbols or one can visually better distinguish between Latin and Greek symbols. Compare e.g. $\mathfrak{a} = a + \alpha$ and $\mathfrak{v} = v + (?\nu)$.
Higher order forms will be in general denoted by $\alpha^{(p)}, \beta^{(q)}, \ldots$ or $\omega^{(p)}, \eta^{(q)}, \rho^{(r)}, \ldots$. There will be exceptions, however , for specific forms like the B-field $B = B_{mn} \mathbf{d}x^m \wedge \mathbf{d}x^n$. Following this logic, we will also denote multivectors (tensors with antisymmetric upper indices) by small letters, indicating their multivector-degree in brackets: $a^{(p)}, b^{(q)}, \ldots$ or $v^{(p)}, w^{(q)}, \ldots$. There are again exceptions, e.g. a Poisson structure will often be denoted by $P = P^{mn} \boldsymbol{\partial}_m \wedge \boldsymbol{\partial}_n$. The most horrible exception is the one of the beta-transformation, which is denoted by a large beta β^{mn} in (C.47), in order to distinguish it from forms.

Tensors of mixed type will be denoted by capital letters where we denote in brackets first the number of lower indices and then the number of upper indices, e.g. $T^{(p,q)}$. Most of the time, we treat multivector valued forms, e.g. the lower indices as well as the upper indices are antisymmetrized. The letters denoting form degree and multivector degree will often be adapted to the letter of the tensor, e.g. $K^{(k,k')}, L^{(l,l')}, \ldots$

Attention: k and l are also used as dummy indices! Sometimes (I'm sorry for that) the same letter appears with different meanings. However, in those situations the dummy indices will carry indices which might even be one of the degrees k or k', e.g. $K_{\ldots}{}^{k_1 \ldots k_{k'}} L_{k_{k'} \ldots k_1 \ldots}{}^{\cdots}$.

Working all the time with graded algebras with a graded symmetric product (the wedge product), everything in this article has to be understood as *graded*. i.e. with commutator we mean the graded commutator and with the Poisson bracket the graded Poisson bracket. They will not be denoted differently than the non-graded operations. Relevant for the sign rules is the *total degree* which we define to be form degree minus the multivector degree. In the field language, it corresponds to the total ghost number which is the pure ghost number minus the antighost number. It will be denoted by

$$| K^{(k,k')} | = k - k' \tag{A.1}$$

As only degrees appear in the exponent of a minus sign, a simplified notation is used there

$$(-)^A \equiv (-1)^{|A|}, \quad (-)^{A+B} \equiv (-)^{|A|+|B|}, \quad (-)^{AB} \equiv (-)^{|A||B|} \quad \forall A, B \tag{A.2}$$

For the Poisson bracket, the following (less common) sign convention is chosen:

$$\{p_m, x^n\} = \delta_m^n = -\{x^n, p_m\} \tag{A.3}$$

$$\{b_m, c^n\} = \delta_m^n = -(-)^{bc} \{c^n, b_m\} \tag{A.4}$$

Derivatives with respect to x^m are denoted by $\frac{\partial}{\partial x^m} f \equiv \partial_m f \equiv f_{,m}$. For graded variables left derivatives are denoted by $\frac{\partial}{\partial \boldsymbol{c}} f(\boldsymbol{c})$, while right derivatives are denoted equivalently by two different notations

$$\partial f(\boldsymbol{c}) / \partial \boldsymbol{c} \equiv f \frac{\overleftarrow{\partial}}{\partial \boldsymbol{c}} \tag{A.5}$$

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The corresponding notations are used for functional derivatives $\frac{\delta}{\delta \mathbf{c}(\sigma)}$. With respect to the wedge product, the basis element $\boldsymbol{\partial}_m$ is an odd object ($\boldsymbol{\partial}_m \wedge \boldsymbol{\partial}_n = -\boldsymbol{\partial}_n \wedge \boldsymbol{\partial}_m$). The partial derivative ∂_k acting on some coefficient function, however, is an even operator (it does not change the parity as long as it is not contracted with a basis element $\mathbf{d}x^k$). That is why we denote the odd basis element $\boldsymbol{\partial}_m$ and $\mathbf{d}x^m$ as well as the odd exterior derivative $\mathbf{d}$ with boldface symbols. The interior product itself does not carry a grading in the sense that $| \imath_K \rho |=| K | + | \rho |$, while for the Lie derivative $\boldsymbol{\mathcal{L}}_K = [\imath_K, \mathbf{d}]$ the $\boldsymbol{\mathcal{L}}$ carries a grading in the sense $| \boldsymbol{\mathcal{L}}_K \rho |=| K | + | \rho | +1$. That is why the Lie derivative is denoted with a boldface $\boldsymbol{\mathcal{L}}$ which is also very good to distinguish it from generalized multivectors $\mathcal{K}, \mathcal{L}, \ldots$. The philosophy of writing odd objects in boldface style is also extended to the combined basis element

$$\mathbf{t}_M \equiv (\boldsymbol{\partial}_m, \mathbf{d}x^m),$$
$$\mathbf{t}^M \equiv (\mathbf{d}x^m, \boldsymbol{\partial}_m) \tag{A.6}$$

and to the comma in the derived bracket $[\,\mathbf{,}\,]$ in contrast to the commutator $[\,,\,]$. This should be, however, just a reminder. It will be obvious for other reasons, which bracket is meant. But we do *not* extend this philosophy to vectors and 1-forms, where it would be consistent (but too much effort) to write the vectors and basis elements in boldface style and the coefficients in standard style. We will instead write the vector in the same style as the coefficient $a = a_m \mathbf{d}x^m$.

A square bracket is used as usual to denote the antisymmetrization of, say p, indices (including a normalization factor $\frac{1}{p!}$). A vertical line is used to exclude some indices from antisymmetrization. An extreme example would be

$$A^{[ab|cd|e|fg|hi]} \tag{A.7}$$

where A is antisymmetrized only in a, b, e, h and i, but not in c, d, f and g. Normally we use only expressions like $A^{[ab|cd|efg]}$, where a, b, e, f and g are antisymmetrized.

Wedge product. A significant difference from usual conventions is that for multivectors, forms and generalized multivectors we include the normalization of the factor already in the definition of the wedge product

$$\mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_n} \equiv \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_n} \equiv \mathbf{d}x^{[m_1} \otimes \ldots \otimes \mathbf{d}x^{m_n]} \equiv \sum_P \frac{1}{n!} \mathbf{d}x^{m_{P(1)}} \otimes \ldots \otimes \mathbf{d}x^{m_{P(n)}} \tag{A.8}$$

$$\boldsymbol{\partial}_{m_1} \cdots \boldsymbol{\partial}_{m_n} \equiv \boldsymbol{\partial}_{m_1} \wedge \cdots \wedge \boldsymbol{\partial}_{m_n} \equiv \boldsymbol{\partial}_{[m_1} \otimes \cdots \otimes \boldsymbol{\partial}_{m_n]} \equiv \sum_P \frac{1}{n!} \boldsymbol{\partial}_{m_{P(1)}} \otimes \cdots \otimes \boldsymbol{\partial}_{m_{P(n)}} \tag{A.9}$$

$$\mathbf{t}_{M_1} \ldots \mathbf{t}_{M_n} \equiv \mathbf{t}_{M_1} \wedge \ldots \wedge \mathbf{t}_{M_n} \equiv \mathbf{t}_{[M_1} \otimes \ldots \otimes \mathbf{t}_{M_n]} \equiv \sum_P \frac{1}{n!} \mathbf{t}_{M_{P(1)}} \otimes \ldots \otimes \mathbf{t}_{M_{P(n)}} \tag{A.10}$$

(where we sum over all permutations P), such that we omit the usual factor of $\frac{1}{p!}$ in the coordinate expression of a p-form, or a p-vector

$$\alpha^{(p)} \equiv \alpha_{m_1 \ldots m_p} \mathbf{d}x^{m_1} \wedge \cdots \wedge \mathbf{d}x^{m_p} \equiv \alpha_{m_1 \ldots m_p} \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_p} \tag{A.11}$$
$$v^{(p)} \equiv v^{m_1 \ldots m_p} \partial_{m_1} \wedge \ldots \wedge \partial_{m_p} \tag{A.12}$$

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Readers who prefer the $\frac{1}{p!}$, can easily reintroduce it in every equation by replacing e.g. the coefficient functions $v^{m_1\ldots m_p} \to \frac{1}{p!}v^{m_1\ldots m_p}$. The equation for the Schouten bracket (B.9), for example, would change as follows:

$$\begin{aligned}\left[v^{(p)},w^{(q)}\right]^{m_1\ldots m_{p+q-1}} &= pv^{[m_1\ldots m_{p-1}|k}\partial_k w^{|m_p\ldots m_{p+q-1}]} \\ &\quad -qv^{[m_1\ldots m_p|}{}_{,k}w^{k\,|m_{p+1}\ldots m_{p+q-1}]} \\ \longrightarrow\ \frac{1}{(p+q-1)!}\left[v^{(p)},w^{(q)}\right]^{m_1\ldots m_{p+q-1}} &= \frac{1}{(p-1)!}\frac{1}{q!}v^{[m_1\ldots m_{p-1}|k}\partial_k w^{|m_p\ldots m_{p+q-1}]} \\ &\quad -\frac{1}{p!}\frac{1}{(q-1)!}v^{[m_1\ldots m_p|}{}_{,k}w^{k\,|m_{p+1}\ldots m_{p+q-1}]}\end{aligned}$$

Schematic index notation. For longer calculations in coordinate form it is useful to introduce the following notation, where every boldface index is assumed to be contracted with the corresponding basis element (at the same position of the index), s.th. the indices are automatically antisymmetrized.

$$\omega^{(p)} = \omega_{m_1\ldots m_p}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_p} \equiv \omega_{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{A.13}$$

$$a^{(p)} = a^{n_1\ldots n_p}\boldsymbol{\partial}_{n_1}\wedge\ldots\boldsymbol{\partial}_{n_p} \equiv a^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{A.14}$$

$$\mathcal{K}^{(p)} = \mathcal{K}_{M_1\ldots M_p}\mathbf{t}^{M_1}\ldots\mathbf{t}^{M_p} \equiv \mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} \tag{A.15}$$

$$= \mathcal{K}^{M_1\ldots M_p}\mathbf{t}_{M_1}\ldots\mathbf{t}_{M_p} \equiv \mathcal{K}^{\boldsymbol{M}\ldots\boldsymbol{M}} \tag{A.16}$$

or for products of tensors e.g.

$$\begin{aligned}\omega_{\boldsymbol{m}\ldots\boldsymbol{m}}\eta_{\boldsymbol{m}\ldots\boldsymbol{m}} &\equiv \omega_{[m_1\ldots m_p}\eta_{m_{p+1}\ldots m_{p+q}]}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_{p+q}} \\ &= \omega_{m_1\ldots m_p}\eta_{m_{p+1}\ldots m_{p+q}}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_{p+q}} = (-)^{pq}\eta_{\boldsymbol{m}\ldots\boldsymbol{m}}\omega_{\boldsymbol{m}\ldots\boldsymbol{m}}\end{aligned}$$

A boldface index might be hard to distinguish from an ordinary one, but this notation is nevertheless easy to recognize, as normally several coinciding indices appear (which are not summed over as they are at the same position). Similarly, for multivector valued forms we define[17]

$$\begin{aligned}&K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\mathbf{d}x^{m_1}\wedge\ldots\wedge\mathbf{d}x^{m_k}\otimes\boldsymbol{\partial}_{m_1}\wedge\ldots\wedge\boldsymbol{\partial}_{m_{k'}} \\ &K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}p}L_{p\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \equiv \\ &\qquad \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'-1}p}L_{pm_1\ldots m_{l-1}}{}^{n_1\ldots n_{l'}}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_{k+l-1}}\otimes\boldsymbol{\partial}_{m_1}\cdots\boldsymbol{\partial}_{m_{k'+l'-1}}\end{aligned} \tag{A.17}$$

[17] Upper and lower signs are thus treated independently. For calculational reasons this is not the best way to do. We can interpret every boldface index on the lefthand side of (A.17) as a basis element sitting at the position of the index, so that the order of the basis elements on the lefthand side is first $k\times\mathbf{d}x^m$, $(k'-1)\boldsymbol{\partial}_m$, $(l-1)\times\mathbf{d}x^m$ and $l'\times\boldsymbol{\partial}_m$, s.th., in order to get the order of the righthand side, we have to interchange $(k'-1)\boldsymbol{\partial}_m$ with $(l-1)\times\mathbf{d}x^m$, which gives a sign factor of $(-)^{(k'-1)(l-1)}$. This is a natural sign factor which appears all the way in the equations, which could be easily absorbed into the definition. However, we wanted to keep the sign factors explicitly in the equations in order to keep the notation as self-explaining as possible and not confuse the reader too much.

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B. Review of geometric brackets as derived brackets

Mathematics in this section is based on the review article on derived brackets by Kosmann-Schwarzbach [1]. The presentation, however, will be somewhat different and in addition to (or sometimes instead of) the abstract definitions coordinate expressions will be given.

B.1 Lie bracket of vector fields, Lie derivative and Schouten bracket

This first subsection is intended to give a feeling, why the Schouten bracket is a very natural extension of the Lie bracket of vector fields. It is a good example to become more familiar with the subject, before we become more general in the subsequent subsections, but it can be skipped without any harm (note however the notation introduced before (B.12)).

Consider the ordinary *Lie-bracket of vector fields* which turns the tangent space of a manifold into a Lie algebra or the tangent bundle into a Lie algebroid and which takes in a local coordinate basis the familiar form

$$[v,w]^m = v^k\partial_k w^m - w^k\partial_k v^m \tag{B.1}$$

We will convince ourselves in the following that numerous other common differential brackets are just natural extensions of this bracket and can be regarded as one and the same bracket. Such a generalized bracket is e.g. useful to formulate integrability conditions and it can serve via the Jacobi identity as a powerful tool in otherwise lengthy calculations . In addition it shows up naturally in some sigma-models as is discussed in section 2.

Given the Lie-bracket of vector fields, it seems natural to extend it to higher rank tensor fields by demanding a Leibniz rule on tensor products of the form $[v,w_1\otimes w_2] = [v,w_1]\otimes w_2 + w_1\otimes[v,w_2]$. Remembering that the Lie-bracket of two vector fields is just the Lie derivative of one vector field with respect to the other

$$[v,w] = \boldsymbol{\mathcal{L}}_v w \tag{B.2}$$

the *Lie derivative* of a general tensor $T = T_{m_1\ldots m_p}^{n_1\ldots n_q}\mathbf{d}x^{m_1}\otimes\ldots\otimes\mathbf{d}x^{m_p}\otimes\boldsymbol{\partial}_{n_1}\otimes\cdots\otimes\boldsymbol{\partial}_{n_q}$ with respect to a vector field v can be seen as a first extension of the Lie bracket:

$$\begin{aligned}[v,T] &\equiv \boldsymbol{\mathcal{L}}_v T \\ [v,T]_{m_1\ldots m_p}^{n_1\ldots n_q} &= v^k\partial_k T_{m_1\ldots m_p}^{n_1\ldots n_q} - \sum_i \partial_k v^{n_i} T_{m_1\ldots m_p}^{n_1\ldots n_{i-1}k\,n_{i+1}\ldots n_q} \\ &\quad + \sum_j \partial_{m_j} v^k T_{m_1\ldots m_{j-1}k\,m_{j+1}\ldots m_p}^{n_1\ldots n_q}\end{aligned} \tag{B.3}$$

The Lie derivative obeys (as a derivative should) the *Leibniz rule*

$$[v,T_1\otimes T_2] = [v,T_1]\otimes T_2 + T_1\otimes[v,T_2] \tag{B.4}$$

In fact, giving as input only the Lie derivative of a scalar ϕ, namely the directional derivative $[v,\phi] \equiv v^k\partial_k\phi$, and the Lie bracket of vector fields (B.1), the Lie derivative of general

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tensors (B.3) is determined by the Leibniz-rule. Insisting on antisymmetry of the bracket, we have to define

$$[T,v] \equiv -[v,T] \tag{B.5}$$

Indeed, it can be checked that the above definitions lead to a valid Jacobi-identity of the form

$$[v,[w,T]] = [[v,w],T] + [w,[v,T]] \quad \text{for arbitrary tensors } T \tag{B.6}$$

which is perhaps better known in the form

$$[\boldsymbol{\mathcal{L}}_v, \boldsymbol{\mathcal{L}}_w]\, T = \boldsymbol{\mathcal{L}}_{[v,w]} T \tag{B.7}$$

We have now vectors acting via the bracket on general tensors, but tensors only acting on vectors via (B.5) . It is thus natural to use Leibniz again to define the action of tensors on tensors. To make a long story short, this is not possible for general tensors. It is possible, however, for tensors with only upper indices which are either antisymmetrized (*multivectors*) or symmetrized (*symmetric multivectors*). We will concentrate in this paper on tensors with antisymmetrized indices (the reason being the natural given differential for forms which also have antisymmetrized indices), but the symmetric case makes perfect sense and at some points we will give short comments. (See e.g. [27] for more information on the Schouten bracket of symmetric tensor fields.)

Given two *multivector fields* (note that the prefactor $1/p!$ is intentionally missing (see page 38).

$$\begin{aligned} v^{(p)} &\equiv v^{m_1\ldots m_p} \boldsymbol{\partial}_{m_1} \wedge \ldots \wedge \boldsymbol{\partial}_{m_p}, \\ w^{(q)} &\equiv w^{m_1\ldots m_q} \boldsymbol{\partial}_{m_1} \wedge \ldots \wedge \boldsymbol{\partial}_{m_q} \end{aligned} \tag{B.8}$$

their Schouten(-Nijenhuis) bracket, or *Schouten bracket* for short, is given in a local coordinate basis by

$$\begin{aligned} \left[v^{(p)},w^{(q)}\right]^{m_1\ldots m_{p+q-1}} &= p v^{[m_1\ldots m_{p-1}|k} \partial_k w^{|m_p\ldots m_{p+q-1}]} \\ &\quad - q v^{[m_1\ldots m_p|}{}_{,k} w^{k\,|m_{p+1}\ldots m_{p+q-1}]} \end{aligned} \tag{B.9}$$

Realizing that the Lie-derivative (B.3) of a multivector field $w^{(q)}$ with respect to a vector $v^{(1)}$ is

$$\left[v,w^{(q)}\right]^{n_1\ldots n_q} = v^k \partial_k w^{n_1\ldots n_q} - q \partial_k v^{[n_1|} w^{k\,|n_2\ldots n_q]} \tag{B.10}$$

one recognizes that (B.9) is a natural extension of this, obeying a Leibniz rule, which we will write down below in (B.17). However, as the coordinate form of generalized brackets will become very lengthy at some point, we will first introduce some *notation* which is more schematic, although still exact. Namely we imagine that every *boldface index* $\boldsymbol{m}$ is an ordinary index m contracted with the corresponding basis vector $\boldsymbol{\partial}_m$ at the position of the index:

$$v^{(p)} = v^{m_1\ldots m_p} \boldsymbol{\partial}_{m_1} \wedge \ldots \wedge \boldsymbol{\partial}_{m_p} \equiv v^{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{B.11}$$

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This saves us the writing of the basis vectors as well as the enumeration or manual antisymmetrization of the indices. As a boldface index might be hard to distinguish from an ordinary one, we will use this notation only for several indices, s.th. we get repeated indices $\boldsymbol{m}\ldots\boldsymbol{m}$ which are easily to recognize (and are not summed over, as they are at the same vertical position). See in the appendix A on page 39 for a more detailed explanation. The Schouten bracket then reads

$$\left[v^{(p)},w^{(q)}\right] = pv^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - qv^{\boldsymbol{m}\ldots\boldsymbol{m}}{}_{,k}w^{k\,\boldsymbol{m}\ldots\boldsymbol{m}} \tag{B.12}$$

$$= pv^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - (-)^{p(q-1)}qw^{k\,\boldsymbol{m}\ldots\boldsymbol{m}}v^{\boldsymbol{m}\ldots\boldsymbol{m}}{}_{,k} \tag{B.13}$$

$$= pv^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - (-)^{(p-1)(q-1)}qw^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k v^{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{B.14}$$

In the last line it becomes obvious that the bracket is *skew-symmetric* in the sense of a Lie algebra of degree[18] -1:

$$\left[v^{(p)},w^{(q)}\right] = -(-)^{(p-1)(q-1)}\left[w^{(q)},v^{(p)}\right] \tag{B.15}$$

It obeys the corresponding *Jacobi identity*

$$\left[v_1^{(p_1)},\left[v_2^{(p_2)},v_3^{(p_3)}\right]\right] = \left[\left[v_1^{(p_1)},v_2^{(p_2)}\right],v_3^{(p_3)}\right] + (-)^{(p_1-1)(p_2-1)}\left[v_2^{(p_2)},\left[v_1^{(p_1)},v_3^{(p_3)}\right]\right] \tag{B.16}$$

Our starting point was to extend the bracket in a way that it acts via Leibniz on the wedge product. A Lie algebra which has a second product on which the bracket acts via Leibniz is known as Poisson algebra. However, here the bracket has degree -1 (it reduces the multivector degree by one) while the wedge product has no degree (the degree of the wedge

[18]A *Lie bracket* $[\ ,_{(n)}\ ]$ *of degree* n in a graded algebra increases the degree (which we denote by $|\ldots|$) by n

$$\left|\left[A,_{(n)} B\right]\right| = |A| + |B| + n$$

It can be understood as an ordinary graded Lie-bracket, when we redefine the grading $\|\ldots\| \equiv |\ldots| + n$, such that the Lie bracket itself does not carry a grading any longer

$$\left\|\left[A,_{(n)} B\right]\right\| = \|A\| + \|B\|$$

The symmetry properties are thus (*skew symmetry of degree* n)

$$\left[A,_{(n)} B\right] = -(-)^{(|A|+n)(|A|+n)}\left[B,_{(n)} A\right]$$

and it obeys the usual graded Jacobi-identity (with shifted degrees)

$$\left[A,_{(n)}\left[B,_{(n)} C\right]\right] = \left[\left[A,_{(n)} B\right],_{(n)} C\right] + (-)^{(|A|+n)(|A|+n)}\left[B,_{(n)}\left[A,_{(n)} C\right]\right]$$

In addition there might be a Poisson-relation with respect to some other product which respects the original grading. To be consistent with both gradings, this relation has to read

$$\left[A,_{(n)} B\cdot C\right] = \left[A,_{(n)} B\right]\cdot C + (-)^{(|A|+n)|B|}B\cdot\left[A,_{(n)} C\right]$$

This is consistent with $B\cdot C = (-)^{|B||C|}C\cdot B$ on the one hand and the skew symmetry of the bracket on the other hand. One can imagine the grading of the bracket to sit at the position of the comma.

For the bracket of multivectors we have as degree the vector degree. Later, when we will have tensors of mixed type (vector and form), we will use the form degree minus the vector degree as total degree. Then the Schouten-bracket is of degree $+1$, which should not confuse the reader.

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product of multivectors is just the sum of the degrees). According to footnote 18, we have to adjust the Leibniz rule. The resulting algebra for Lie brackets of degree -1 is known as *Gerstenhaber algebra* or in this special case *Schouten algebra* (which is the standard example for a Gerstenhaber algebra). The *Leibniz rule* is

$$\left[v_1^{(p_1)}, v_2^{(p_2)} \wedge v_3^{(p_3)}\right] = \left[v_1^{(p_1)}, v_2^{(p_2)}\right] \wedge v_3^{(p_3)} + (-)^{(p_1-1)p_2} v_2^{(p_2)} \wedge \left[v_1^{(p_1)}, v_3^{(p_3)}\right] \quad \text{(B.17)}$$

The standard example in field theory for a Poisson algebra is the phase space equipped with the Poisson bracket or the commutator of operators or matrices.[19] The Schouten algebra is naturally realized by the *antibracket* of the BV antifield formalism (see subsection 2.5).

B.2 Embedding of vectors into the space of differential operators

The Leibniz rule is not the only concept to generalize the vector Lie bracket to higher rank tensors. The major difficulty in the definition of brackets between higher rank tensors is the Jacobi-identity, which should hold for them. It is therefore extremely useful to have a mechanism which automatically guarantees the Jacobi identity. A way to get such a mechanism is to *embed* the tensors into some space of differential operators, as for the operators we have the commutator as natural Lie bracket which might in turn induce some bracket on the tensors we started with. Vector fields e.g. naturally act on differential forms via the *interior product*

$$\imath_v \omega^{(p)} \equiv p \cdot v^k \omega_{k\boldsymbol{m}\ldots\boldsymbol{m}} \quad \text{(B.18)}$$

This can be seen as the embedding of vector fields in the space of differential operators acting on forms, because the interior product with respect to a vector is a graded derivative with the grading -1 of the vector (we take as total degree the form degree minus the multivector degree, which for a vector is just -1)

$$\imath_v \left(\omega^{(p)} \wedge \eta^{(q)}\right) = \imath_v \omega^{(p)} \wedge \eta^{(q)} + (-)^q \omega^{(p)} \wedge \imath_v \eta^{(q)} \quad \text{(B.19)}$$

Taking the idea of above we can take the commutator of two interior products. We note, however, that it only induces a trivial (always vanishing) bracket on the vectorfields

$$[\imath_v, \imath_w] = 0 = \imath_0 \quad \text{(B.20)}$$

As the interior product (B.18) does not include any partial derivative on the vector-coefficient, it was clear from the beginning that this ansatz does not lead to the Lie bracket of vector fields or any generalization of it. We have to bring the exterior derivative into the game, in our notation

$$\mathbf{d}\omega^{(p)} = \partial_{\boldsymbol{m}} \omega_{\boldsymbol{m}\ldots\boldsymbol{m}} \quad \text{(B.21)}$$

There are two ways to do this

[19] In fact, working with totally symmetric multivector fields would have lead to a Poisson algebra instead of a Gerstenhaber algebra.

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- *Change the embedding:* Instead of embedding the vectors via the interior product acting on forms, we can embed them via the Lie-derivative acting on forms. When acting on forms, the Lie derivative can be written as the (graded) commutator of interior product and exterior derivative

$$\boldsymbol{\mathcal{L}}_v = [\imath_v, \mathbf{d}] \qquad \boldsymbol{\mathcal{L}}_v \omega^{(p)} = v^k \partial_k \omega_{\boldsymbol{m}\ldots\boldsymbol{m}} + p \cdot \partial_{\boldsymbol{m}} v^k \omega_{k\,\boldsymbol{m}\ldots\boldsymbol{m}} \tag{B.22}$$

  Indeed, using the Lie derivative as embedding $v \mapsto \boldsymbol{\mathcal{L}}_v$, the commutator of Lie derivatives induces the Lie bracket of vector fields (a special case of (B.7)

$$[\boldsymbol{\mathcal{L}}_v, \boldsymbol{\mathcal{L}}_w] = \boldsymbol{\mathcal{L}}_{[v,w]} \tag{B.23}$$

- *Change the bracket:* In the space of differential operators acting on forms, the commutator is the most natural Lie bracket. However, the existence of a nilpotent odd operator acting on our algebra, namely the commutator with the exterior derivative, enables the construction of what is called a *derived bracket.*[20]

$$[\imath_v,_{\mathbf{d}} \imath_w] \equiv [[\imath_v, \mathbf{d}], \imath_w] \tag{B.24}$$

  This derived bracket (which is in this case a Lie bracket again, as we are considering the abelian subalgebra of interior products of vector fields) indeed induces the Lie bracket of vector fields when we use the interior product as embedding

$$[\imath_v,_{\mathbf{d}} \imath_w] = \imath_{[v,w]} \tag{B.25}$$

[20] Given a bracket $[\ ,_{(n)}\ ]$ of degree n (not necessarily a Lie bracket. It can be as well a *Loday bracket* where the skew-symmetry property as compared to footnote 18 is missing, but the Jacobi identity still holds) and a differential D (derivation of degree 1 and square 0), its *derived bracket* [28, 29, 1] (which is of degree $n+1$) is defined by

$$[a,_{(\mathrm{D})} b] = (-)^{n+a+1} [\mathrm{D}a,_{(n)} b]$$

We put the subscript (D) at the position of the comma, to indicate that the grading of D is sitting there. The strange sign is just to make the definition nicer for the most frequent case of an interior derivation, where $\mathrm{D}a = [d,_{(n)} a]$ with d some element of the algebra with degree $\mid d \mid = 1-n$ and $[d,_{(n)} d] = 0$, s.th. we have

$$[a,_d b] = [[a,_{(n)} d],_{(n)} b]$$

The derived bracket is then again a Loday bracket (of degree $n+1$) and obeys the corresponding Jacobi-identity (that is always the nontrivial part). If a, b are elements of a commuting subalgebra ($[a,_{(n)} b] = 0$), the derived bracket even is skew-symmetric and thus a Lie bracket of degree $n+1$.

In the case at hand we start with a Lie bracket of degree 0 (the commutator) and take as interior derivation the commutator with the exterior derivative $[\mathbf{d}, \ldots]$. Note that the exterior derivative itself is a derivative on forms, but not on the space of differential operators on forms. Therefore we need the commutator.

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The above equations plus two additional ones are the well known *Cartan formulae*

$$[\imath_v, \imath_w] = 0 = [\mathbf{d}, \mathbf{d}] \tag{B.26}$$

$$\boldsymbol{\mathcal{L}}_v = [\imath_v, \mathbf{d}] \tag{B.27}$$

$$[\boldsymbol{\mathcal{L}}_v, \mathbf{d}] = 0 \tag{B.28}$$

$$[\boldsymbol{\mathcal{L}}_v, \boldsymbol{\mathcal{L}}_w] = \boldsymbol{\mathcal{L}}_{[v,w]} \tag{B.29}$$

$$\big[\underbrace{[\imath_v, \mathbf{d}]}_{\boldsymbol{\mathcal{L}}_v}, \imath_w]\big] = \imath_{[v,w]} \tag{B.30}$$

(B.23) can be rewritten, using Jacobi's identity and $[\mathbf{d}, \mathbf{d}] = 0$, as

$$[[[\imath_v, \mathbf{d}], \imath_w], \mathbf{d}] = \left[\imath_{[v,w]}, \mathbf{d}\right] \tag{B.31}$$

Starting from (B.25), one thus arrives at (B.23) by simply taking the commutator with $\mathbf{d}$. We will therefore concentrate in the following on the second possibility, using the derived bracket, as the first one can be deduced from it. Let us just mention that the generalization in the spirit of the derived bracket (B.25) (or more precise its skew-symmetrization) is known as *Vinogradov bracket* [30, 31] (see footnote 25), while the generalization in the spirit of (B.23) is known as *Buttin's bracket* [23].

B.3 Derived bracket for multivector valued forms

Let us now consider a much more general case, namely the space of multivector valued forms, i.e. tensors which are antisymmetric in the upper as well as in the lower indices. With the Schouten bracket we have a bracket for multivectors, which are antisymmetric in all (upper) indices. There exists as well a bracket for vector valued forms, namely tensors with one upper index and arbitrary many antisymmetrized lower indices. This bracket (which we have not yet discussed) is the (Fröhlicher-) Nijenhuis bracket (see (B.64)), which shows up in the integrability condition for almost complex structures. Multivector valued forms have arbitrary many antisymmetrized upper and arbitrary antisymmetrized lower indices and thus contain both cases. The antisymmetrization appears quite naturally in field theory (we give only a few remarks about completely symmetric indices, which appear as well, but which will not be subject of this paper). It makes also sense to define brackets on sums of tensors of different type (e.g. the Dorfman bracket for generalized complex geometry). Those brackets are then simply given by linearity.

So let us consider two vector valued forms (we denote the number of lower indices and

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the number of upper indices in this order via superscripts)[21]

$$K^{(k,k')} \equiv K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}$$
$$\equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_k}\otimes\boldsymbol{\partial}_{n_1}\cdots\boldsymbol{\partial}_{n_{k'}} \quad \text{(B.32)}$$
$$L^{(l,l')} \equiv L_{\underbrace{\boldsymbol{m}\ldots\boldsymbol{m}}_{l}}{}^{\overbrace{\boldsymbol{n}\ldots\boldsymbol{n}}^{}}{}_{\scriptstyle l'} \quad \text{(B.33)}$$

Note the use of the schematic index notation, which we used for upper indices already in subsection B.1 and which is explained in the appendix A on page 39. Following the ideas of above, we want to embed those vector valued forms in some space of differential operators. As we have upper as well as lower indices now, it is less clear why we should choose the space of operators acting on forms and not on some other tensors for the embedding. However, the space of forms is the only one where we have a natural exterior derivative without using any extra structure.[22] Therefore we will define again a natural embedding into the space of differential operators acting on forms as a generalization of the interior product. Namely, we will act with a multivector valued form K on a form ρ by just contracting all upper indices with form-indices and antisymmetrizing the remaining lower indices s.th. we get again a form as result. The formal definition goes in two steps. First one defines the interior product with multivectors. For a decomposable multivector $v^{(p)} = v_1 \wedge \ldots \wedge v_p$ set

$$\imath_{v_1\wedge\ldots\wedge v_p}\rho^{(r)} \equiv \imath_{v_1}\cdots\imath_{v_p}\rho^{(r)} \quad \text{(B.34)}$$

This fixes the interior product for a generic multivector uniquely (contracting all indices with form-indices). The next step is to define for a multivector valued form $K^{(k,k')} = \eta^{(k)} \wedge v^{(k')}$ which is decomposable in a form and a multivector, that it acts on a form by first acting with the multivector as above and then wedging the result with the form

$$\imath_{\eta^{(k)}\wedge v^{(k')}}\rho \equiv \eta^{(k)} \wedge \imath_{v^{(k)}}\rho = (-)^{k'k}\imath_{v^{(k')}\wedge\eta^{(k)}}\rho \quad \text{(B.35)}$$

It is kind of a normal ordering that $\imath_{v^{(k')}}$ acts first:

$$\imath_\eta\imath_v = \imath_{\eta^{(k)}\wedge v^{(k')}} = (-)^{kk'}\imath_{v^{(k')}\wedge\eta^{(k)}} \neq \imath_v\imath_\eta \quad \text{(B.36)}$$

For a generic multivector valued form, the above definitions fix the following coordinate form of the *interior product*[23] with a multivector valued form

$$\imath_{K^{(k,k')}}\rho^{(r)} \equiv (k')!\binom{r}{k'}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}}\rho_{\underbrace{l_{k'}\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}_{r}} \quad \text{(B.37)}$$

[21]One can certainly map a tensor $K_m{}^n\mathbf{d}x^m\otimes\boldsymbol{\partial}_n$ to one where the basis elements are antisymmetrized $K_m{}^n\mathbf{d}x^m\wedge\boldsymbol{\partial}_n \overset{\text{see page 38}}{\equiv} \frac{1}{2}K_m{}^n\mathbf{d}x^m\otimes\boldsymbol{\partial}_n - \frac{1}{2}K_m{}^n\boldsymbol{\partial}_n\otimes\mathbf{d}x^m$ and vice versa. In the field theory applications we will always get a complete antisymmetrization. This mapping is the reason why we take care for the horizontal positions of the indices. It should just indicate the order of the basis elements which was chosen for the mapping.

[22]One can define an exterior derivative — the *Lichnerowicz-Poisson differential* — on the space of multivectors as well (via the Schouten bracket), but for this we need an integrable Poisson structure: $\mathbf{d}_P N^{(q)} \equiv \left[P^{(2)}, N^{(q)}\right]$, with $\left[P^{(2)}, P^{(2)}\right] = 0$

[23]The name 'interior product' is misleading in the sense that the operation is (for decomposable tensors) a composition of interior and exterior wedge product. It will, however, in the generalizations of Cartan's

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So we are just contracting all the upper indices of K with an appropriate number of indices of the form and are wedging the remaining lower indices. The origin of the combinatorial prefactor is perhaps more transparent in the phase space formulation (2.13) in subsection 2.1. For multivectors $v^{(p)}$ and $w^{(q)}$ the operator product of $\imath_{v^{(p)}}$ and $\imath_{w^{(q)}}$ induces, due to (B.34) simply the wedge product of the multivectors

$$\imath_{v^{(p)}}\imath_{w^{(q)}} = \imath_{v^{(p)}\wedge w^{(q)}} \tag{B.38}$$

But for general multivector-valued forms we have instead[24]

$$\begin{aligned} \imath_{K^{(k,k')}}\imath_{L^{(l,l')}} &= \sum_{p=0}^{k'} \imath_{\imath_K^{(p)} L} \\ &= \imath_{K\wedge L} + \sum_{p=1}^{k'} \imath_{\imath_K^{(p)} L} \end{aligned} \tag{B.39}$$

with

$$\imath^{(p)}_{K^{(k,k')}} L^{(l,l')} \equiv (-)^{(k'-p)(l-p)} p! \begin{pmatrix} k' \\ p \end{pmatrix} \begin{pmatrix} l \\ p \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} L_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{B.40}$$

For $p = k'$, $\imath^{(p)}_K$ reduces to the interior product (B.37). Both are in general not a derivative any longer. $\imath^{(p)}$ is, however, a p-th order derivative, as contracting p indices means taking the p-th derivative with respect to p basis elements (see 2.18 in subsection 2.1). Our embedding $\imath_{K^{(k,k')}}$ in (B.37) is therefore a k'-th order derivative. For $p = 0$ on the other hand, $\imath^{(p)}_K$ is just a wedge product with K

While for vectors the commutator of two interior products (B.20) did only induce a trivial bracket on vectors, which is the same for multivectors due to (B.38), this is different

formulae play the role of the interior product. We will therefore stick to this name. We can also see it as a short name for 'interior product of maximal order' in the sense that all upper indices are contracted as opposed to an interior 'product of order p', where we contract only p upper indices. 'Order' is in the sense of the order of a derivative. While $\imath_v$ is a derivative for any vector v, the general interior product acts like a higher order derivative.

[24]The product of interior products in (B.39) induces a noncommutative product for the multivector-valued forms, whose commutator is the algebraic bracket, namely

$$\begin{aligned} K * L &\equiv \sum_{p\geq 0} \imath^{(p)}_K L \\ [K, L]^{\Delta} &= K * L - (-)^{(k-k')(l-l')} L * K \end{aligned}$$

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for multivector-valued forms.

$$\left[\imath_{K^{(k,k')}}, \imath_{L^{(l,,l')}}\right] = \imath_{[K,L]^{\Delta}} \tag{B.41}$$

$$[K,L]^{\Delta} \equiv \sum_{p\geq 1} \underbrace{\imath_K^{(p)} L - (-)^{(k-k')(l-l')} \imath_L^{(p)} K}_{\equiv [K,L]^{\Delta}_{(p)}} \tag{B.42}$$

$$\begin{aligned} &= \sum_{p\geq 1} (-)^{(k'-p)(l-p)} p! \begin{pmatrix} k' \\ p \end{pmatrix} \begin{pmatrix} l \\ p \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} L_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \\ &\quad -(-)^{(k-k')(l-l')} (-)^{(l'-p)(k-p)} p! \begin{pmatrix} l' \\ p \end{pmatrix} \begin{pmatrix} k \\ p \end{pmatrix} \times \\ &\quad \times L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} K_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \end{aligned} \tag{B.43}$$

where we introduced an *algebraic bracket* $[K,L]^{\Delta}$ in the second line, which is is due to Buttin [23], and which is a generalization of the Nijenhuis-Richardson bracket for vector-valued forms (B.60). As it was induced via the embedding from the graded commutator, it has the same properties, i.e. it is graded antisymmetric and obeys the graded Jacobi identity. Actually, the term with lowest p, so $[K,L]^{\Delta}_{(p=1)}$, is itself an algebraic bracket, which appears in subsection 2.1.1 as canonical Poisson bracket. It is known under the name *Buttin's algebraic bracket* ([23], denoted in [1] by $[\ ,\ ]^0_B$) or as *big bracket*

$$\begin{aligned} [K,L]^{\Delta}_{(1)} = \imath_K^{(1)} L - (-)^{(k-k')(l-l')} \imath_L^{(1)} K &= (-)^{(k'-1)(l-1)} k' l \cdot K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1} L_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \\ &\quad -(-)^{(k-k')(l-l')} (-)^{(l'-1)(k-1)} l' k \cdot L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1} K_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \end{aligned} \tag{B.44}$$

But as for the vector fields in subsection B.2, we are rather interested in the derived bracket of $[K,L]^{\Delta}$, or at the bracket induced via an embedding based on the Lie derivative. An obvious generalization of the Lie derivative is the commutator $[\imath_K, \mathbf{d}]$, which will be a derivative of the same order as $\imath_K$ and therefore is not a derivative in the sense that it obeys the Leibniz rule. Although it is common to use this generalization, I am not aware of an appropriate name for it. Let us just call it the *Lie derivative with respect to K* (being a derivative of order k')

$$\boldsymbol{\mathcal{L}}_{K^{(k,k')}} \equiv \left[\imath_{K^{(k,k')}}, \mathbf{d}\right] \tag{B.45}$$

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{K^{(k,k')}} \rho &= (k')! \begin{pmatrix} r+1 \\ k' \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \partial_{[l_{k'}} \rho_{l_{k'-1}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}]} + \\ &\quad -(-)^{k-k'} (k')! \begin{pmatrix} r \\ k' \end{pmatrix} \partial_{\boldsymbol{m}} \left(K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \rho_{l_{k'}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}} \right) \end{aligned} \tag{B.46}$$

$$\begin{aligned} &= (k')! \begin{pmatrix} r \\ k'-1 \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \partial_{l_{k'}} \rho_{l_{k'-1}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}} + \\ &\quad -(-)^{k-k'} (k')! \begin{pmatrix} r \\ k' \end{pmatrix} \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \rho_{l_{k'}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}} \end{aligned} \tag{B.47}$$

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The Lie derivative above is an ingredient to calculate the *derived bracket* (remember footnote 20 on page 44) which is given by[25]

$$[\imath_K,_{\mathbf{d}}\imath_L] \equiv [[\imath_K, \mathbf{d}], \imath_L] \equiv \imath_{[K,L]} \quad \text{if possible} \tag{B.48}$$

One should distinguish the derived bracket on the level of operators on the left from the derived bracket on the tensors $[K,L]$ on the right. Only in special cases the result of the commutator on the left can be written as the interior product of another tensorial object which then can be considered as the derived bracket with respect to the algebraic bracket $[\,,\,]^{\Delta}$. Therefore one normally does not find an explicit general expression for this derived bracket in literature. In 2.1.2, however, the meaning of exterior derivative and interior product are extended in order to be able to write down an explicit general coordinate expression (2.48) which reduces in the mentioned special cases to the well known results (see e.g. B.4.2).

Closely related to the derived bracket in (B.48) of above is *Buttin's differential bracket*, given by

$$[\boldsymbol{\mathcal{L}}_K, \boldsymbol{\mathcal{L}}_L] \equiv \boldsymbol{\mathcal{L}}_{[K,L]_B} \quad \text{if possible} \tag{B.49}$$

Because of $[\mathbf{d}, \mathbf{d}] = 0$ and $\boldsymbol{\mathcal{L}}_K = [\imath_K, \mathbf{d}]$ we have (using Jacobi)

$$[\boldsymbol{\mathcal{L}}_K, \boldsymbol{\mathcal{L}}_L] = [[\imath_K,_{\mathbf{d}}\imath_L], \mathbf{d}] = [[\imath_K,_{\mathbf{d}}\imath_L], \mathbf{d}] \stackrel{!}{=} [\imath_{[K,L]_B}, \mathbf{d}] \tag{B.50}$$

Comparing with (B.48) s.th. in cases where $[K,L]$ exists, the brackets have to coincide up to a closed term, or locally a total derivative

$$\imath_{[K,L]} = \imath_{[K,L]_B} + [\mathbf{d}, \ldots] \tag{B.51}$$

Using again the extended definition of exterior derivative and interior product of 2.1.2, this relation can be rewritten as

$$[K,L] = [K,L]_B + \mathbf{d}(\ldots) \tag{B.52}$$

The Nijenhuis bracket (B.71) is the major example for this relation.

B.4 Examples

B.4.1 Schouten(-Nijenhuis) bracket

Let us shortly review the Schouten bracket under the new aspects. For multivectors $v^{(p)}, w^{(q)}$ the algebraic bracket vanishes

$$[\imath_{v^{(p)}}, \imath_{w^{(q)}}] = 0 \tag{B.53}$$

[25] The *Vinogradov bracket* [31, 30] (see also [1]) is a bracket in the space of all graded endomorphisms in the space of differential forms $\Omega^\bullet(M)$

$$[a,b]_V = \frac{1}{2}\Big([[a,d], b] - (-)^b [a, [b, d]]\Big) \quad \forall a, b \in \Omega^\bullet(M)$$

It is the skew symmetrization of a derived bracket. The embedding of the multivector valued forms into the endomorphisms $\Omega^\bullet(M)$ via the interior product which we consider is neither closed under the Vinogradov bracket nor under the derived bracket in the general case.

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The *Schouten bracket* $\left[v^{(p)},w^{(q)}\right]$ coincides with the derived bracket as well as with Buttin's differential bracket, i.e. we have

$$[[\imath_{v^{(p)}}, \mathbf{d}], \imath_{w^{(q)}}] = \imath_{\left[v^{(p)},w^{(q)}\right]} \tag{B.54}$$
$$[\boldsymbol{\mathcal{L}}_{v^{(p)}}, \boldsymbol{\mathcal{L}}_{w^{(q)}}] = \boldsymbol{\mathcal{L}}_{\left[v^{(p)},w^{(q)}\right]} \tag{B.55}$$

Its coordinate form — given already before in (B.14) – is

$$\left[v^{(p)},w^{(q)}\right] = pv^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - (-)^{(p-1)(q-1)}qw^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k v^{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{B.56}$$

The vector Lie bracket is a special case of the Schouten bracket as well as of the Nijenhuis bracket.

B.4.2 (Fröhlicher-)Nijenhuis bracket and its relation to the Richardson-Nijenhuis bracket

Consider vector valued forms, i.e. tensors of the form

$$K^{(k,1)} \equiv K_{m_1\ldots m_k}{}^n \mathbf{d}x^{m_1} \wedge \cdots \wedge \mathbf{d}x^{m_k} \wedge \boldsymbol{\partial}_n \cong K_{m_1\ldots m_k}{}^n \mathbf{d}x^{m_1} \wedge \cdots \wedge \mathbf{d}x^{m_k} \otimes \boldsymbol{\partial}_n \tag{B.57}$$

The algebraic bracket of two such tensors, defined via the graded commutator (note that $\mid \imath_K \mid=\mid K \mid= k-1$)

$$[\imath_K, \imath_L] = \imath_{[K,L]^\Delta} \tag{B.58}$$

consists only of the first term in the expansion, because we have only one upper index to contract.

$$\left[K^{(k,1)}, L^{(l,1)}\right]^\Delta = \left[K^{(k,1)}, L^{(l,1)}\right]_{(1)}^\Delta = \imath_K^{(1)} L - (-)^{(k-1)(l-1)}\imath_L^{(1)} K = \tag{B.59}$$
$$\stackrel{(B.44)}{==} l\, K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j L_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} - (-)^{(k-1)(l-1)} k\, L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} \tag{B.60}$$

It is thus just the big bracket or Buttin's algebraic bracket but in this case it is known as *Richardson-Nijenhuis-bracket.*

The Lie derivative of a form with respect to K (in the sense of (B.45)) is because of $k' = 1$ really a (first order) derivative and takes the form

$$\boldsymbol{\mathcal{L}}_{K^{(k,1)}} \equiv [\imath_{K^{(k,1)}}, \mathbf{d}] \tag{B.61}$$
$$\boldsymbol{\mathcal{L}}_{K^{(k,1)}}\rho^{(r)} = K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^l \partial_l \rho_{\boldsymbol{m}\ldots\boldsymbol{m}} + (-)^k r \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^l \rho_{l\boldsymbol{m}\ldots\boldsymbol{m}} \tag{B.62}$$

The *(Froehlicher-)Nijenhuis* bracket is defined as the unique tensor $[K,L]_N$, s.th.

$$[\boldsymbol{\mathcal{L}}_K, \boldsymbol{\mathcal{L}}_L] = \boldsymbol{\mathcal{L}}_{[K,L]_N} \tag{B.63}$$

It is therefore an example of Buttin's differential bracket. Its explicit coordinate form reads

$$[K,L]_N \equiv K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j \partial_j L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + (-)^k l \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j L_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} +$$
$$-(-)^{kl} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j \partial_j K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} - (-)^{kl}(-)^l k \partial_{\boldsymbol{m}} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} \tag{B.64}$$
$$= \text{"}\boldsymbol{\mathcal{L}}_K L - (-)^{kl}\boldsymbol{\mathcal{L}}_L K\text{"} \tag{B.65}$$

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A different point of view on the Nijenhuis bracket is via the *derived bracket* on the level of the differential operators acting on forms:

$$[\imath_K ,_{\mathbf{d}} \imath_L] \equiv [[\imath_K, \mathbf{d}] , \imath_L] \tag{B.66}$$

It induces the Nijenhuis-bracket only up to a total derivative (the Lie-derivative-term)

$$[\imath_K ,_{\mathbf{d}} \imath_L] \equiv \imath_{[K,L]_N} - (-)^{k(l-1)} \mathcal{L}_{\imath_L K} \tag{B.67}$$

Using the extended definition of the exterior derivative in the sense of (2.35) and of the interior product (2.31), one can write the Lie derivative as an interior product (see 2.33) $\mathcal{L}_{\imath_L K} = -(-)^{l+k} \imath_{\mathbf{d}(\imath_L K)}$ and $[[\imath_K, \mathbf{d}] , \imath_L] = (-)^k [\imath_{\mathbf{d}K}, \imath_L] = (-)^k \imath_{[\mathbf{d}K,L]^\Delta}$, so that we can rewrite (B.67) as

$$[K , L] \equiv [K , L]_N + (-)^{(k-1)l} \mathbf{d}(\imath_L K) \tag{B.68}$$

$$\text{with } [K , L] \equiv (-)^k [\mathbf{d}K, L]^\Delta \tag{B.69}$$

In that sense, $[K , L]$ is the derived bracket of the Richardson Nijenhuis bracket while the Nijenhuis bracket differs by a total derivative. The explicit coordinate form can be read off from (2.46), (2.48) (with only the $p = 1$ term surviving)

$$[K , L] = (-)^k \imath^{(1)}_{\mathbf{d}K} L + (-)^{kl} (-)^l \imath^{(1)}_{\mathbf{d}L} K + (-)^{(k-1)l} \mathbf{d}(\imath^{(p)}_L K) = \tag{B.70}$$

$$\begin{aligned} &= K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j \partial_j L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + (-)^k l \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j L_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + \\ &\quad - (-)^{kl} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j \partial_j K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} - (-)^{kl} (-)^l k \partial_{\boldsymbol{m}} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + \\ &\quad + (-)^{(k-1)l} \mathbf{d}\big(\underbrace{k L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}}}_{\imath_L K} \big) \end{aligned} \tag{B.71}$$

where the last part is non-tensorial due to the appearance of the basis element p_i (see subsection 2.1.2):

$$\mathbf{d}(\imath_L K) = \mathbf{d}\big(k L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}}\big) = k \partial_{\boldsymbol{m}} \left(L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} \right) - (-)^{l+k} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^i p_i \tag{B.72}$$

The remaining part coincides with the coordinate form of the *Nijenhuis bracket* as given in (B.64).

One can nicely summarize the algebra of graded derivations on forms as

$$\begin{aligned} &\left[\mathcal{L}_{K_1^{(k_1)}} + \imath_{L_1^{(l_1)}} , \mathcal{L}_{K_2^{(k_2)}} + \imath_{L_2^{(l_2)}} \right] = \\ &= \mathcal{L}_{[K_1,K_2]_N + \imath_{L_1} K_2 - (-)^{(l_2-1)k_1} \imath_{L_2} K_1} + \imath_{[K_1,L_2]_N - (-)^{(l_1-1)k_2} [K_2,L_1]_N + [L_1,L_2]^\Delta} \end{aligned} \tag{B.73}$$

C. Some aspects of generalized (complex) geometry

For introductions into Hitchin's [5] generalized complex geometry (GCG) see e.g. Zabzine's review [16] or Gualtieri's thesis [3]. For a survey of compactification with fluxes and its relation to GCG see Graña's review [9].

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C.1 Basics

In *generalized geometry* one is looking at structures (e.g. a complex structure) on the direct sum of tangent and cotangent bundle $T \oplus T^*$. Let us call a section of this bundle a *generalized vector* (field) or synonymously *generalized 1-form*, which is the sum of a vector field and a 1-form

$$\begin{aligned} \mathfrak{a} &= a + \alpha && \text{(C.1)} \\ &= a^m \boldsymbol{\partial}_m + \alpha_m \mathbf{d}x^m && \text{(C.2)} \end{aligned}$$

Using the *combined basis elements*

$$\mathbf{t}_M \equiv (\boldsymbol{\partial}_m, \mathbf{d}x^m) \tag{C.3}$$

a generalized vector $\mathfrak{a}$ can be written as

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}^M \mathbf{t}_M && \text{(C.4)} \\ \mathfrak{a}^M &= (a^m, \alpha_m) && \text{(C.5)} \end{aligned}$$

There is a *canonical metric* $\mathcal{G}$ on $T \oplus T^*$

$$\begin{aligned} \langle \mathfrak{a}, \mathfrak{b} \rangle &\equiv \alpha(b) + \beta(a) && \text{(C.6)} \\ &= \alpha_m b^m + \beta_m a^m && \text{(C.7)} \\ &\equiv \mathfrak{a}^M \mathcal{G}_{MN} \mathfrak{b}^N && \text{(C.8)} \end{aligned}$$

with

$$\mathcal{G}_{MN} \equiv \begin{pmatrix} 0 & \delta_m^n \\ \delta_n^m & 0 \end{pmatrix} \tag{C.9}$$

which has *signature* (d,-d) (if d is the dimension of the base manifold). The above definition differs by a factor of 2 from the most common one. We prefer, however, to have an inverse metric of the same form

$$\mathcal{G}^{MN} \equiv \left(\mathcal{G}^{-1}\right)^{MN} = \begin{pmatrix} 0 & \delta_n^m \\ \delta_m^n & 0 \end{pmatrix} \tag{C.10}$$

As it is constant, we can always pull it through partial derivatives. Using this metric to lower and raise indices just interchanges vector and form component. We can equally rewrite $\mathfrak{a}$ in (C.4) with a basis with upper capital indices and the vector coefficients with lower indices

$$\begin{aligned} \mathbf{t}^M &\equiv (\mathbf{d}x^m, \boldsymbol{\partial}_m) && \text{(C.11)} \\ \mathfrak{a} &= \mathfrak{a}_M \mathbf{t}^M && \text{(C.12)} \\ \mathfrak{a}_M &= (\alpha_m, a^m) && \text{(C.13)} \end{aligned}$$

Note that in the present paper there is no existence of any metric on the tangent bundle assumed. Therefore we cannot raise or lower small indices. In cases where 1-form and

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vector have a similar symbol, the position of the small index therefore uniquely determines which is which (e.g. ω_m and w^m).

In addition to the canonical metric $\mathcal{G}_{MN}$ there is also a *canonical antisymmetric 2-form* $\mathcal{B}$, s.th. $\alpha(b) - \beta(a) = \mathfrak{a}^M \mathcal{B}_{MN} \mathfrak{b}^N$ with coordinate form

$$\mathcal{B}_{MN} \equiv \begin{pmatrix} 0 & -\delta_m^n \\ \delta_n^m & 0 \end{pmatrix} \tag{C.14}$$

Raising the indices with $\mathcal{G}^{MN}$ yields

$$\mathcal{B}^M{}_N = \begin{pmatrix} \delta_n^m & 0 \\ 0 & -\delta_m^n \end{pmatrix} = -B_N{}^M \tag{C.15}$$

$$\mathcal{B}^{MN} = \begin{pmatrix} 0 & \delta_n^m \\ -\delta_m^n & 0 \end{pmatrix} \tag{C.16}$$

We can thus use $\mathcal{B}$ and $\mathcal{G}$ to construct *projection operators* $\mathcal{P}_\mathcal{T}$ and $\mathcal{P}_{\mathcal{T}^*}$ to tangent and cotangent space

$$\mathcal{P}_\mathcal{T}{}^M{}_N \equiv \frac{1}{2}\left(\delta^M{}_N + B^M{}_N\right) = \begin{pmatrix} \delta_n^m & 0 \\ 0 & 0 \end{pmatrix} \tag{C.17}$$

$$\mathcal{P}_{\mathcal{T}^*}{}^M{}_N \equiv \frac{1}{2}\left(\delta^M{}_N - B^M{}_N\right) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_m^n \end{pmatrix} \tag{C.18}$$

$$\mathcal{P}_\mathcal{T}\mathfrak{a} = a, \qquad \mathcal{P}_{\mathcal{T}^*}\mathfrak{a} = \alpha \tag{C.19}$$

C.2 Generalized almost complex structure

A *generalized almost complex structure* is a linear map from $T \oplus T^*$ to itself which squares to minus the identity-map, i.e. in components

$$\mathcal{J}^M{}_K \mathcal{J}^K{}_N = -\delta_N^M \tag{C.20}$$

It is called a *generalized complex structure* if it is integrable (see subsection C.4). It should be *compatible* with our canonical metric $\mathcal{G}$ which means that it should behave like multiplication with i in a Hermitian scalar product of a complex vector space[26]

$$\langle \mathfrak{v}, \mathcal{J}\mathfrak{w} \rangle = -\langle \mathcal{J}\mathfrak{v}, \mathfrak{w} \rangle \iff (\mathcal{G}\mathcal{J})^T = -\mathcal{G}\mathcal{J} \iff \mathcal{J}_{MN} = -\mathcal{J}_{NM} \tag{C.21}$$

This property is also known as *antihermiticity* of $\mathcal{J}$. Because of (C.21), $\mathcal{J}$ can be written as

$$\mathcal{J}^M{}_N = \begin{pmatrix} J^m{}_n & P^{mn} \\ -Q_{mn} & -J^n{}_m \end{pmatrix} \qquad \mathcal{J}_{MN} = \begin{pmatrix} -Q_{mn} & -J^n{}_m \\ J^m{}_n & P^{mn} \end{pmatrix} \tag{C.22}$$

[26] In a complex vector space with Hermitian scalar product $\langle a, b\rangle = \overline{\langle b, a\rangle}$ we have $\langle a, ib\rangle = -\langle ia, b\rangle$.

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where P^{mn} and Q_{mn} are antisymmetric matrices, and (C.20) translates into

$$J^2 - PQ = -\mathbb{1} \tag{C.23}$$
$$JP - PJ^T = 0 \tag{C.24}$$
$$-QJ + J^T Q = 0 \tag{C.25}$$

Here it becomes obvious that the generalized complex structure contains the case of an ordinary almost complex structure J with $J^2 = -1$ for $Q = P = 0$ as well as the case of an almost symplectic structure of a non-degenerate 2-form Q with existing inverse $PQ = \mathbb{1}$ for $J = 0$. In addition to those algebraic constraints, the integrability of the generalized almost complex structure gives further differential conditions (see subsection C.4) which boil down in the two special cases to the integrability of the ordinary complex structure or to the integrability of the symplectic structure.

Because of $\mathcal{J}^2 = -\mathbb{1}$, $\mathcal{J}$ has eigenvalues $\pm i$. The corresponding eigenvectors span the space of *generalized holomorphic vectors* L or generalized antiholomorphic vectors $\bar{L}$ respectively. This provides a natural splitting of the complexified bundle

$$(T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L} \tag{C.26}$$

The *projector* Π to the space of eigenvalue $+i$ (namely L) can be be written as

$$\Pi \equiv \frac{1}{2}\left(\mathbb{1} - i\mathcal{J}\right) \tag{C.27}$$

while the projector to $\bar{L}$ is just the complex conjugate $\bar{\Pi} = \frac{1}{2}\left(\mathbb{1} + i\mathcal{J}\right) = G^{-1}\Pi^T G$. Indeed, for any generalized vector field $\mathfrak{v}$ we have

$$\mathcal{J}\Pi\mathfrak{v} = i\Pi\mathfrak{v} \tag{C.28}$$

L and $\bar{L}$ are what one calls *maximally isotropic subspaces*, i.e. spaces which are *isotropic*

$$\langle \mathfrak{v}, \mathfrak{w} \rangle = 0 \quad \forall \mathfrak{v}, \mathfrak{w} \in L \tag{C.29}$$

(this is because $\Pi^T G \Pi = \mathcal{G}\bar{\Pi}\Pi = 0$) and which have half the dimension of the complete bundle. As the canonical metric $\langle \cdots \rangle$ is nondegenerate, this is the maximal possible dimension for isotropic subbundles.

C.3 Dorfman and Courant bracket

Something which seems to be a bit unnatural in this whole business in the beginning is the introduction of the Courant bracket, which is the antisymmetrization of the so-called Dorfman-bracket. The *Dorfman bracket* in turn is the natural generalization of the Lie

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bracket from the point of view of derived brackets (B.48)[27]

$$[[\imath_{\mathfrak{a}}, \mathbf{d}], \imath_{\mathfrak{b}}] = \imath_{[\mathfrak{a},\mathfrak{b}]} \tag{C.30}$$

$$\text{where } [\mathfrak{a},\mathfrak{b}] \equiv [a,b] + \mathcal{L}_a\beta - \mathcal{L}_b\alpha + \mathbf{d}(\imath_b\alpha) \tag{C.31}$$

$$= [a,b] + \mathcal{L}_a\beta - \imath_b(\mathbf{d}\alpha) \tag{C.32}$$

$$= \mathcal{L}_a\mathfrak{b} - \imath_b(\mathbf{d}\alpha) \tag{C.33}$$

To get a homogeneous coordinate expression, we define

$$\partial_M \equiv (\partial_m, 0) \quad \Rightarrow \partial^M = (0, \partial_m) \tag{C.34}$$

The Dorfman bracket can then be written as[28]

$$[\mathfrak{a},\mathfrak{b}]^M = \mathfrak{a}^K\partial_K\mathfrak{b}^M + \left(\partial^M\mathfrak{a}_K - \partial_K\mathfrak{a}^M\right)\mathfrak{b}^K \tag{C.35}$$

$$\text{or } [\mathfrak{a},\mathfrak{b}]_M = \mathfrak{a}^K\partial_K\mathfrak{b}_M + 2\partial_{[M}\mathfrak{a}_{K]}\mathfrak{b}^K \tag{C.36}$$

Apart from the term in the middle $\partial^M\mathfrak{a}_K$, (C.35) looks formally the same as the Lie bracket of vector fields (B.1). The Dorfman bracket is in general not antisymmetric but it obeys a

[27] The twisted Dorfman bracket is defined similarly via

$$[[\imath_{\mathfrak{a}}, \mathbf{d} + H\wedge], \imath_{\mathfrak{b}}] \equiv \imath_{[\mathfrak{a},\mathfrak{b}]_H}$$

Remembering that $H\wedge = \imath_H$ and using $[\imath_a, \imath_H] = \imath_{[a,H]^\Delta} = \imath_{\imath_a^{(1)}H}$, we get

$$[\mathfrak{a},\mathfrak{b}]_H \equiv [a,b] - \imath_b\imath_a H$$

[28] It is perhaps interesting to note that this notation of the partial derivative with capital index suggests the extension to a derivative with respect to some dual coordinate

$$\partial^m \equiv \partial_{\tilde{x}_m}$$

We could understand this as coordinates of a dual manifold whose tangent space coincides in some sense with the cotangent space of the original space and vice versa. This might be connected to Hull's doubled geometry [32–35].

To see that such an ad-hoc extension of the Dorfman bracket is not completely unfounded, note that there is a more general notion of a Dorfman bracket (or Courant bracket) in the context of Lie-bialgebroids (for a definition see e.g. [3, p.32,20]). There we have two Lie algebroids L and L^* which are dual with respect to some inner product and which both carry some Lie bracket. (For T and T^*, only T carries a Lie bracket in the beginning. For a non-trivial Lie bracket of forms on T^* we need some extra structure like e.g. a Poisson structure which would lead to the Koszul bracket on forms.) The Lie bracket on L induces a differential $\mathbf{d}$ on L^* and the Lie bracket on L^* induces a differential $\mathbf{d}^*$ on L. The definition for the Dorfman bracket on the Lie bialgebroid $L \oplus L^*$ is then

$$[\mathfrak{a},\mathfrak{b}] \equiv [a,b] + \mathcal{L}_a\beta - \mathcal{L}_b\alpha + \mathbf{d}(\imath_b\alpha) + \\ + [\alpha,\beta] + \mathcal{L}_\alpha b - \mathcal{L}_\beta a + \mathbf{d}^*(\imath_\beta a)$$

The first line is the part we are used to from our usual Dorfman bracket on $T \oplus T^*$, while second line is the corresponding part coming from the nontrivial structure on L^*. Taking now $L = T$, $L^* = T^*$ and assuming that $[\alpha,\beta]$ and $\mathcal{L}_\alpha$ and $\mathbf{d}^*$ are a Lie bracket, Lie derivative and exterior derivative built in the ordinary way, but with the new partial derivative w.r.t. the dual coordinates ∂^m, the coordinate form of the Dorfman bracket remains exactly the one of (C.35), (C.36), but with $\partial_M = (\partial_m, 0)$ replaced by $\partial_M = (\partial_m, \partial^m)$.

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Jacobi-identity (Leibniz from the left) of the form

$$[\mathfrak{a}, [\mathfrak{b},\mathfrak{c}]] = [[\mathfrak{a},\mathfrak{b}],\mathfrak{c}] + [\mathfrak{b}, [\mathfrak{a},\mathfrak{c}]] \tag{C.37}$$

Although the Dorfman bracket is all we need, most of the literature on generalized complex geometry so far works with its antisymmetrization, which is called *Courant bracket*

$$[\mathfrak{a},\mathfrak{b}]_- \equiv [a,b] + \mathcal{L}_a\beta - \mathcal{L}_b\alpha + \frac{1}{2}\mathbf{d}(\imath_b\alpha - \imath_a\beta) \tag{C.38}$$

$$[\mathfrak{a},\mathfrak{b}]_{-M} = \mathfrak{a}^K\partial_K\mathfrak{b}_M - \partial_K\mathfrak{a}_M\mathfrak{b}^K + \frac{1}{2}\left(\partial_M\mathfrak{a}_K\mathfrak{b}^K - \mathfrak{a}^K\partial_M\mathfrak{b}_K\right) \tag{C.39}$$

and which does not obey any Jacobi identity. As it is much simpler to go from Dorfman to Courant, than the other way round, we will only work with the Dorfman bracket. On any isotropic subspace ($\imath_b\alpha + \imath_a\beta = 0$) the two coincide anyway, i.e. they become a Lie bracket, obeying Jacobi and being antisymmetric.

We call a transformation a *symmetry of the bracket* when the bracket of two vectors transforms in the same way as the vectors

$$[(\mathfrak{b}+\delta\mathfrak{b}),(\mathfrak{c}+\delta\mathfrak{c})] = [\mathfrak{b},\mathfrak{c}] + \delta[\mathfrak{b},\mathfrak{c}] \tag{C.40}$$

$$\delta[\mathfrak{b},\mathfrak{c}] = [\delta\mathfrak{b},\mathfrak{c}] + [\mathfrak{b},\delta\mathfrak{c}] + [\delta\mathfrak{b},\delta\mathfrak{c}] \tag{C.41}$$

i.e. infinitesimal symmetry transformations (where the last term drops) have to obey a product rule. Similar as for the Lie-bracket of vector fields, infinitesimal transformations are generated by the bracket itself. Let us call the corresponding derivative, in analogy to the Lie derivative, the *Dorfman derivative* of a generalized vector with respect to a generalized vector.

$$\delta\mathfrak{b} = \mathcal{D}_\mathfrak{a}\mathfrak{b} \equiv [\mathfrak{a},\mathfrak{b}] \tag{C.42}$$

These transformations are therefore, due to the Jacobi-identity (C.37) always symmetries of the bracket. From (C.33) we can see that the Dorfman derivative consists of a usual Lie derivative and second part which acts only on the vector part of $\mathfrak{b}$ by contracting it with the exact 2-form $\mathbf{d}\alpha$

$$\mathcal{D}_a\mathfrak{b} = \mathcal{L}_a\mathfrak{b} \tag{C.43}$$

$$\mathcal{D}_\alpha b = -\imath_b(\mathbf{d}\alpha) = b^m(\partial_n\alpha_m - \partial_m\alpha_n)\mathbf{d}x^n \tag{C.44}$$

In fact, it is enough for the 2-form to be closed, in order to get a symmetry. If we replace $-\mathbf{d}\alpha$ by a *closed 2-form* B, the transformation is known as B*-transform*

$$\delta_B b = \imath_b B \tag{C.45}$$

Finally, we should note that the B-transform is part of the $O(d,d)$-transformations, i.e. the transformations which leave the canonical metric invariant. As usual for orthogonal groups the infinitesimal generators are antisymmetric when the second index is pulled down

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with the corresponding metric. The generators of an $O(d,d)$-transformation can therefore be written as [3, p.6]

$$\Omega_{MN} = \begin{pmatrix} B_{mn} & -A_m{}^n \\ A_n{}^m & \beta^{mn} \end{pmatrix} \tag{C.46}$$

$$\Omega^M{}_N = \begin{pmatrix} A_n{}^m & \beta^{mn} \\ B_{mn} & -A_m{}^n \end{pmatrix} \tag{C.47}$$

In addition to the B-transform, acting with Ω on a generalized vector induces the so-called *beta-transform* on the 1-form component[29] as well as $Gl(d)$-transformations of vector and 1-form component via A. For constant tensors, the Lie-derivative is just a $Gl(d)$ transformation. Therefore both symmetries of the Dorfman bracket are symmetries of the canonical metric $\mathcal{G}$ as well. For this reason the canonical metric is invariant under the *Dorfman derivative* $\boldsymbol{\mathcal{D}}_{\mathfrak{v}}$with respect to a generalized vector $\mathfrak{v}$, which we define on generalized rank p tensors using (C.35) in a way that it acts via Leibniz on tensor products (like the Lie derivative) and as a directional derivative on scalars

$$(\boldsymbol{\mathcal{D}}_{\mathfrak{v}}\mathcal{T})^{M_1\ldots M_p} \equiv \mathfrak{v}^K\partial_K\mathcal{T}^{M_1\ldots M_p} + \sum_i(\partial^{M_i}\mathfrak{v}_K - \partial_K\mathfrak{v}^{M_i})\mathcal{T}^{M_1\ldots M_{i-1}KM_{i+1}\ldots M_p} \tag{C.48}$$

$$\boldsymbol{\mathcal{D}}_{\mathfrak{v}}(\mathcal{A}\otimes\mathcal{B}) = \boldsymbol{\mathcal{D}}_v\mathcal{A}\otimes\mathcal{B} + \mathcal{A}\otimes\boldsymbol{\mathcal{D}}_v\mathcal{B} \tag{C.49}$$

$$\boldsymbol{\mathcal{D}}_{\mathfrak{v}}(\phi) = \mathfrak{v}^K\partial_K\phi = v^k\partial_k\phi \tag{C.50}$$

Acting on the canonical metric, one recovers the fact, that the Dorfman derivative contains the isometries of the metric

$$\boldsymbol{\mathcal{D}}_{\mathfrak{v}}\mathcal{G} = 2(\partial^{M_1}\mathfrak{v}_K - \partial_K\mathfrak{v}^{M_1})\mathcal{G}^{KM_2} = 0 \tag{C.51}$$

Comparing the role of Lie-derivative and Dorfman-derivative, the B-transform should be understood as an extension of diffeomorphisms. In string theory it shows up in the Buscher-rules for T-duality ([36, 37]) and can perhaps be better understood geometrically via Hull's doubled geometry [32 − 34] (compare to footnote 28). The beta-transform is not a symmetry of the Dorfman bracket as it stands. However, if we introduce dual coordinates as suggested in footnote 28, the beta-transform would show up in the symmetry-transformations of the extended Dorfman bracket generated by itself.[30]

[29] The letter β for the beta-transformations does not really fit into the philosophy of the present notations, where we use small Greek letters for 1-forms (or sometimes p-forms) only, but not for multivectors. As the transformation is, however, commonly known as beta-transformation, we use a large β, in order to distinguish it from the one-forms β, which are floating around.

[30] Taking the Dorfman bracket of footnote 28, we get as Dorfman derivative of a generalized vector $\mathfrak{c}$ instead of (C.43), (C.44) the extended transformation

$$\boldsymbol{\mathcal{D}}_a\mathfrak{c} \equiv \boldsymbol{\mathcal{L}}_a\mathfrak{c} - \imath_\gamma(\mathbf{d}^*a)$$
$$\boldsymbol{\mathcal{D}}_\alpha\mathfrak{c} \equiv -(\imath_c\mathbf{d}\alpha) + \boldsymbol{\mathcal{L}}_\alpha\mathfrak{c}$$

i.e. the first line is extended by a beta-transformation of γ with $\beta = -\mathbf{d}^*a$ and the B-transform of α ($B = -\mathbf{d}\alpha$) in the second line is extended by a Lie derivative with respect to α.

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On an isotropic subspace L (e.g. the generalized holomorphic subspace) Courant- and Dorfman-bracket coincide and have the properties of a Lie bracket. It is therefore possible to define a Schouten bracket on generalized multivectors on $\bigwedge^{\bullet} L$ which have e.g. only generalized holomorphic indices (compare [3, p.21]). If we use again the notation with repeated boldface indices

$$\mathcal{A}^{(p)} \equiv \mathcal{A}_{\boldsymbol{M}\ldots\boldsymbol{M}} \equiv \mathcal{A}_{M_1\ldots M_p}\mathbf{t}^{M_1}\cdots\mathbf{t}^{M_2} \tag{C.52}$$

we get as coordinate form for this *Dorfman-Schouten bracket*

$$\left[\mathcal{A}^{(p)},\mathcal{B}^{(q)}\right] = p\mathcal{A}^{\boldsymbol{M}\ldots\boldsymbol{M}K}\partial_K\mathcal{B}^{\boldsymbol{M}\ldots\boldsymbol{M}} + q\left(p\partial^{\boldsymbol{M}}\mathcal{A}_K{}^{\boldsymbol{M}\ldots\boldsymbol{M}} - \partial_K\mathcal{A}^{\boldsymbol{M}\ldots\boldsymbol{M}}\right)\mathcal{B}^{K\boldsymbol{M}\ldots\boldsymbol{M}} \tag{C.53}$$

In the first term in the bracket on the righthand side, the $\partial^{\boldsymbol{M}}$ can as well be shifted with a minus sign to $\mathcal{B}$, because in $\bigwedge^{\bullet} L$ we have only isotropic indices in the sense that

$$\mathcal{A}^{\boldsymbol{M}\ldots\boldsymbol{M}}{}_K\mathcal{B}^{K\boldsymbol{M}\ldots\boldsymbol{M}} = 0 \tag{C.54}$$

For this reason, the Dorfman-Schouten bracket has really the required skew-symmetry of a Schouten-bracket

$$\left[\mathcal{A}^{(p)},\mathcal{B}^{(q)}\right] = -(-)^{(q+1)(p+1)}\left[\mathcal{B}^{(q)},\mathcal{A}^{(p)}\right] \tag{C.55}$$

On $\bigwedge^{\bullet} L$ this bracket coincides with the derived bracket of the big bracket, as the extra term with p_M in (C.69) vanishes because of (C.54).

C.4 Integrability

Integrability for an ordinary complex structure means that there exist in any chart $\dim_M/2$ holomorphic vector fields (with respect to the almost complex structure) which can be integrated to holomorphic coordinates z^a in this chart of the manifold and make it a complex manifold. Those vector fields are then just $\partial/\partial z^a$. Those coordinate differentials have vanishing Lie bracket among each other (partial derivatives commute). In turn, every set of vectors with vanishing Lie bracket can be integrated to coordinates. The existence of such a set of integrable holomorphic vector fields is guaranteed when the holomorphic subbundle is closed under the Lie bracket, i.e. the Lie bracket of two holomorphic vector fields is again a holomorphic vector field.

As the Dorfman bracket restricted to the generalized holomorphic subbundle $L \subset (T \oplus T^*) \otimes \mathbb{C}$ has the properties of a Lie bracket, we can demand exactly the same for generalized holomorphic vectors as above for holomorphic ones. The condition for the generalized complex structure to be integrable is thus that the generalized holomorphic subbundle L is closed under the Dorfman bracket, i.e. in terms of the projectors

$$\bar{\Pi}\left[\Pi\mathfrak{v},\Pi\mathfrak{w}\right] = 0 \tag{C.56}$$

$$\Longleftrightarrow \quad [\mathfrak{v},\mathfrak{w}] - [\mathcal{J}\mathfrak{v},\mathcal{J}\mathfrak{w}] + \mathcal{J}\left[\mathcal{J}\mathfrak{v},\mathfrak{w}\right] + \mathcal{J}\left[\mathfrak{v},\mathcal{J}\mathfrak{w}\right] = 0 \tag{C.57}$$

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In the following two sub-subsections we will show that this is equivalent to the vanishing of a *generalized Nijenhuis-tensor* [3, p.25] of the coordinate form[31,32]

$$\frac{1}{4}\mathcal{N}^{M_1M_2M_3} \equiv \mathcal{J}^{[M_1|K}\partial_K\mathcal{J}^{|M_2M_3]} + \mathcal{J}^{[M_1|K}\mathcal{J}_K{}^{|M_2,M_3]} \stackrel{!}{=} 0 \tag{C.58}$$

Recalling that

$$\begin{aligned} \mathcal{J}^{MN} &= \begin{pmatrix} P^{mn} & J^m{}_n \\ -J^n{}_m & -Q_{mn} \end{pmatrix}, \\ \mathcal{J}_M{}^N &= \begin{pmatrix} -J^n{}_m & -Q_{mn} \\ P^{mn} & J^m{}_n \end{pmatrix}, \\ \partial^M &= (0, \partial_m) \end{aligned} \tag{C.59}$$

we can rewrite this condition in ordinary tensor components, just to compare it with the conditions given in literature (for the antisymmetrization of the capital indices we take into account that in the last term of (C.58) the indices M_1 and M_2 are automatically

[31]This looks formally like the generalized Schouten bracket (e.g. [3, p.21]) on $\bigwedge^\bullet L$ (with L being the generalized holomorphic bundle) of $\mathcal{J}$ with itself (see also the statement below (C.69)), but it is not, as $\mathcal{J}$ has neither holomorphic nor antiholomorphic indices

$$\begin{aligned} \Pi\mathcal{J} &= i\Pi \neq \mathcal{J} \\ \bar{\Pi}\mathcal{J} &= -i\Pi \neq \mathcal{J} \end{aligned}$$

In fact, we get zero if we contract both indices with the holomorphic projector

$$\Pi^N{}_L\Pi^M{}_K\mathcal{J}^{KL} = \Pi\mathcal{J}\Pi^T = i\Pi\bar{\Pi} = 0$$

The same happens for two antiholomorphic projectors. But we can project one index with an holomorphic projector and the other one with an antiholomorphic one. This yields

$$\bar{\Pi}^N{}_L\Pi^M{}_K\mathcal{J}^{KL} = \Pi\mathcal{J}\Pi = i\Pi$$

Up to a constant prefactor the bracket of Π with Π coincides with the bracket of $\mathcal{J}$ with $\mathcal{J}$. And like for the ordinary complex structure, where we have the Nijenhuis bracket of the complex structure with itself, which has one index in T and the second in T^*, we could here take Π with one index in L and the other in $\bar{L}$ and regard the bracket as generalized Nijenhuis bracket of Π with itself.

[32]If instead the twisted Dorfman bracket (see footnote 27) is used, one gets the integrability condition for a twisted generalized complex structure with a twisted generalized Nijenhuis tensor. Consider the closed three form $H = H_{M_1M_2M_3}\mathbf{t}^{M_1}\mathbf{t}^{M_2}\mathbf{t}^{M_3}$ with $H_{m_1m_2m_3}$ the only nonvanishing components. The twisted generalized Nijenhuis tensor then reads

$$\mathcal{N}^H_{M_1M_2M_3} = \mathcal{N}_{M_1M_2M_3} + 6H_{M_1M_2M_3} - 18\mathcal{J}_{M_1}{}^K H_{KM_2L}\mathcal{J}^L{}_{M_3}$$

Like (C.60)–(C.61) this twisted generalized Nijenhuis tensor as well matches with the tensors given in [19] if one redefines $H_{mnk} \to \frac{1}{3!}H_{mnk}$.

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antisymmetrized because of $\mathcal{J}^2 = -1$):

$$\begin{aligned}
\frac{1}{4}\mathcal{N}^{m_1m_2m_3} &= P^{[m_1|k}\partial_k P^{|m_2m_3]} && \text{(C.60)}\\
&\overset{!}{=} 0\\
\frac{1}{4}\mathcal{N}_n{}^{m_1m_2} &= \frac{1}{3}\left(-J^k{}_n\partial_k P^{[m_1m_2]} + 2P^{[m_1|k}\partial_k J^{|m_2]}{}_n - P^{[m_1|k}J^{|m_2]}{}_{k,n} + J^{[m_1|}{}_k P^{k|m_2]}{}_{,n}\right)\\
&\overset{!}{=} 0\\
\frac{1}{4}\mathcal{N}^n{}_{m_1m_2} &= \frac{1}{3}\left(-P^{nk}\partial_k Q_{[m_1m_2]} + 2J^k{}_{[m_1|}\partial_k J^n{}_{|m_2]} + 2J^n{}_k J^k{}_{[m_1,m_2]} - 2P^{nk}Q_{k[m_1,m_2]}\right)\\
&\overset{!}{=} 0\\
\frac{1}{4}\mathcal{N}_{m_1m_2m_3} &= J^k{}_{[m_1|}\partial_k Q_{|m_2m_3]} + J^k{}_{[m_1|}Q_{k|m_2,m_3]} - Q_{[m_1|k}J^k{}_{|m_2,m_3]}\\
&\overset{!}{=} 0 && \text{(C.61)}
\end{aligned}$$

If we compare those expressions with the tensors A, B, C and D given in (2.16) of [19, p.7], we recognize (replacing Q by $-Q$) that our first line is just $\frac{1}{3}A$, the second line is $-\frac{1}{3}B$ (using (C.24)), the third $\frac{1}{3}C$ and the fourth line is $-\frac{1}{3}D$. There, in turn, it is claimed that the expressions are equivalent to those originally given in (3.16)–(3.19) of [15, p.7].

C.4.1 Coordinate based way to derive the generalized Nijenhuis-tensor

In this sub-subsection we will see that calculations with capital-index notation is rather convenient. So we simply calculate (C.57) brute force by using the explicit coordinate formula for the Dorfman-bracket

$$[\mathfrak{v},\mathfrak{w}]^M = \mathfrak{v}^K\partial_K\mathfrak{w}^M + \left(\partial^M\mathfrak{v}_K - \partial_K\mathfrak{v}^M\right)\mathfrak{w}^K \qquad \text{(C.62)}$$

The brackets of interest are:

$$\begin{aligned}
[\mathfrak{v},\mathcal{J}\mathfrak{w}]^N &= \mathfrak{v}^K\partial_K\mathcal{J}^N{}_L\mathfrak{w}^L + \mathcal{J}^N{}_L\mathfrak{v}^K\partial_K\mathfrak{w}^L + \left(\partial^N\mathfrak{v}_K - \partial_K\mathfrak{v}^N\right)(\mathcal{J}\mathfrak{w})^K\\
(\mathcal{J}[\mathfrak{v},\mathcal{J}\mathfrak{w}])^M &= \underline{\mathfrak{v}^K\mathcal{J}^M{}_N\partial_K\mathcal{J}^N{}_L\mathfrak{w}^L} - \mathfrak{v}^K\partial_K\mathfrak{w}^M + \mathcal{J}^M{}_N\left(\partial^N\mathfrak{v}_K - \partial_K\mathfrak{v}^N\right)(\mathcal{J}\mathfrak{w})^K\\
[\mathcal{J}\mathfrak{v},\mathfrak{w}]^N &= \mathcal{J}^K{}_L\mathfrak{v}^L\partial_K\mathfrak{w}^N + \left(\partial^N\mathcal{J}_{KL} - \partial_K\mathcal{J}^N{}_L\right)\mathfrak{v}^L\mathfrak{w}^K + \left(\mathcal{J}_K{}^L\partial^N\mathfrak{v}_L - \mathcal{J}^N{}_L\partial_K\mathfrak{v}^L\right)\mathfrak{w}^K\\
(\mathcal{J}[\mathcal{J}\mathfrak{v},\mathfrak{w}])^M &= \mathcal{J}^M{}_N(\mathcal{J}\mathfrak{v})^K\partial_K\mathfrak{w}^N + \underline{\mathcal{J}^M{}_N\left(\partial^N\mathcal{J}_{KL} - \partial_K\mathcal{J}^N{}_L\right)\mathfrak{v}^L\mathfrak{w}^K}\\
&\quad -(\mathcal{J}\mathfrak{w})^L\mathcal{J}^M{}_N\partial^N\mathfrak{v}_L + \partial_K\mathfrak{v}^M\mathfrak{w}^K\\
[\mathcal{J}\mathfrak{v},\mathcal{J}\mathfrak{w}]^M &= \mathcal{J}^K{}_N\mathfrak{v}^N\partial_K\mathcal{J}^M{}_L\mathfrak{w}^L + \mathcal{J}^K{}_N\mathfrak{v}^N\mathcal{J}^M{}_L\partial_K\mathfrak{w}^L\\
&\quad + \left(\partial^M\mathcal{J}_{KN}\mathfrak{v}^N - \partial_K\mathcal{J}^M{}_N\mathfrak{v}^N\right)\mathcal{J}^K{}_L\mathfrak{w}^L\\
&\quad + \left(\mathcal{J}_{KN}\partial^M\mathfrak{v}^N - \mathcal{J}^M{}_N\partial_K\mathfrak{v}^N\right)\mathcal{J}^K{}_L\mathfrak{w}^L\\
&= (\mathcal{J}\mathfrak{v})^K\mathcal{J}^M{}_L\partial_K\mathfrak{w}^L - \mathcal{J}^M{}_N\partial_K\mathfrak{v}^N(\mathcal{J}\mathfrak{w})^K\\
&\quad + \underline{\left(\mathcal{J}^K{}_L\partial^M\mathcal{J}_{KN} + 2\mathcal{J}^K{}_{[N|}\partial_K\mathcal{J}^M{}_{|L]}\right)\mathfrak{v}^N\mathfrak{w}^L} + \partial^M\mathfrak{v}_L\mathfrak{w}^L && \text{(C.63)}
\end{aligned}$$

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The underlined terms sum up in the complete expression to the generalized Nijenhuis tensor, while the rest cancels

$$
\begin{aligned}
0 &\overset{!}{=} [\mathfrak{v},\mathfrak{w}]^M - [\mathcal{J}\mathfrak{v},\mathcal{J}\mathfrak{w}]^M + (\mathcal{J}[\mathcal{J}\mathfrak{v},\mathfrak{w}])^M + (\mathcal{J}[\mathfrak{v},\mathcal{J}\mathfrak{w}])^M \\
&= \left(2\mathcal{J}^M{}_K\partial_{[N}\mathcal{J}^K{}_{L]} - \mathcal{J}^K{}_L\partial^M\mathcal{J}_{KN} + \mathcal{J}^{MK}\partial_K\mathcal{J}_{LN} - 2\mathcal{J}^K{}_{[N|}\partial_K\mathcal{J}^M{}_{|L]}\right)\mathfrak{v}^N\mathfrak{w}^L \\
&= \mathfrak{v}_N\left(3\mathcal{J}^{[M|}{}_K\mathcal{J}^{K|L,N]} + 3\mathcal{J}^{[N|K}\partial_K\mathcal{J}^{|ML]}\right)\mathfrak{w}_L \\
&= \frac{3}{4}\mathfrak{v}_N\mathcal{N}^{NML}\mathfrak{w}_L
\end{aligned}
\tag{C.64}
$$

C.4.2 Derivation via derived brackets

Eventually we want to see directly how the generalized Nijenhuis tensor is connected to derived brackets. We will use our insight from the subsections 2.1.1 and 2.1.2. Remember, our basis $\mathbf{t}^M = (\mathbf{d}x^m, \boldsymbol{\partial}_m)$ was identified with the conjugate (ghost-)variables $\mathbf{t}^M \equiv (\boldsymbol{c}^m, \boldsymbol{b}_m)$. One can define generalized multi-vector fields of the form

$$
\mathcal{K}^{(\mathrm{K})} \equiv \mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} \equiv \mathcal{K}_{M_1\ldots M_{\mathrm{K}}}\mathbf{t}^{M_1}\cdots\mathbf{t}^{M_{\mathrm{K}}}
\tag{C.65}
$$

They are in fact just sums of multivector valued forms:

$$
\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} = \sum_{k=0}^{\mathrm{K}}\binom{\mathrm{K}}{k}\mathcal{K}_{\underbrace{\boldsymbol{m}\ldots\boldsymbol{m}}_{k}}{}^{\overbrace{\boldsymbol{n}\ldots\boldsymbol{n}}^{\mathrm{K}-k}} \equiv \sum_{k=0}^{\mathrm{K}} K^{(k,\mathrm{K}-k)}
\tag{C.66}
$$

The big bracket, or Buttin's algebraic bracket is then just the canonical Poisson bracket

$$
[\mathcal{K},\mathcal{L}]^{\Delta}_{(1)} \equiv \mathrm{KL}\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\mathcal{L}_{I\boldsymbol{M}\ldots\boldsymbol{M}} = \{\mathcal{K},\mathcal{L}\}
\tag{C.67}
$$

$$
\{\mathbf{t}_M,\mathbf{t}_N\} = \mathcal{G}_{MN}
\tag{C.68}
$$

The coordinate expression for its derived bracket (compare to (2.49), (2.51)) reads

$$
\begin{aligned}
(-)^{\mathrm{K}-1}\left[\mathbf{d}\mathcal{K}^{(\mathrm{K})},\mathcal{L}^{(\mathrm{L})}\right]^{\Delta}_{(1)} &= \mathrm{K}\cdot\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\partial_I\mathcal{L}_{\boldsymbol{M}\ldots\boldsymbol{M}} - (-)^{(\mathrm{K}+1)(\mathrm{L}+1)}\mathrm{L}\cdot\mathcal{L}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\partial_I\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} \\
&\quad + (-)^{\mathrm{K}-1}\mathrm{KL}\partial_{\boldsymbol{M}}\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\mathcal{L}_{I\boldsymbol{M}\ldots\boldsymbol{M}} + \mathrm{K}(\mathrm{K}-1)\,\mathrm{L}\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{IJ}\mathcal{L}_{I\boldsymbol{M}\ldots\boldsymbol{M}}p_J
\end{aligned}
\tag{C.69}
$$

with $p_J \equiv (p_j, 0)$ and $\partial_I \equiv (\partial_i, 0)$. In the case were both $\mathcal{K}$ and $\mathcal{L}$ only have generalized holomorphic indices, the p-term drops and this expression should coincide with the Schouten-bracket on $\bigwedge^{\bullet} L$ for the holomorphic Lie-algebroid L (see e.g. [3, p.21] and footnote 31). For two rank-two objects, like the generalized complex structure $\mathcal{J}$, this reduces to

$$
[\mathcal{K},_{\mathbf{d}}\mathcal{L}]^{\Delta}_{(1)} = 2\cdot\mathcal{K}_{\boldsymbol{M}}{}^{I}\partial_I\mathcal{L}_{\boldsymbol{M}\boldsymbol{M}} + 2\cdot\mathcal{L}_{\boldsymbol{M}}{}^{I}\partial_I\mathcal{K}_{\boldsymbol{M}\boldsymbol{M}} - 4\partial_{\boldsymbol{M}}\mathcal{K}_{\boldsymbol{M}}{}^{I}\mathcal{L}_{I\boldsymbol{M}} + 4\mathcal{K}^{IJ}\mathcal{L}_{I\boldsymbol{M}}p_J
\tag{C.70}
$$

which reads for two coinciding tensors $\mathcal{J}$

$$
[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} = 4\cdot\mathcal{J}_{\boldsymbol{M}}{}^{I}\partial_I\mathcal{J}_{\boldsymbol{M}\boldsymbol{M}} - 4\partial_{\boldsymbol{M}}\mathcal{J}_{\boldsymbol{M}}{}^{I}\mathcal{J}_{I\boldsymbol{M}} - 4\mathcal{J}^{JI}\mathcal{J}_{I\boldsymbol{M}}p_J
\tag{C.71}
$$

$$
\underset{\mathcal{J}^2=-1}{\overset{(C.58)}{=}} \mathcal{N}_{\boldsymbol{M}\ldots\boldsymbol{M}} + 4\underbrace{p_{\boldsymbol{M}}\mathbf{t}^{M}}_{=\boldsymbol{o}\ (2.8)}
\tag{C.72}
$$

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where $\boldsymbol{o} = \mathbf{d}x^k p_k = -\mathbf{d}(\mathbf{d}x^k \wedge \boldsymbol{\partial}_k)$. We will verify this relation between the generalized Nijenhuis tensor and the derived bracket in the following calculation, where we calculate $\mathcal{N}$ using the big bracket (C.67) all the time. This bracket is like a matrix multiplication if one of the objects has only one index. We will use this fact frequently for the multiplication of $\mathcal{J}$ with a vector

$$\mathcal{J}\mathfrak{v} \equiv \mathcal{J}^M{}_N \mathfrak{v}^N \mathbf{t}_M = \frac{1}{2}\{\mathcal{J},\mathfrak{v}\} \tag{C.73}$$

$$\Rightarrow \{\mathcal{J},\{\mathcal{J},\mathfrak{v}\}\} = 4\mathcal{J}^2\mathfrak{v} = -4\mathfrak{v} = \{\{\mathfrak{v},\mathcal{J}\},\mathcal{J}\} \tag{C.74}$$

$$\{\{\mathfrak{v},\mathcal{J}\},\{\mathcal{J},\mathfrak{w}\}\} = -4\mathfrak{v}^K\mathfrak{w}_K = -4\{\mathfrak{v},\mathfrak{w}\} \tag{C.75}$$

If both objects are of higher rank, however, antisymmetrization of the remaining indices modifies the result. We thus have to be careful with the following examples

$$\{\mathcal{J},\mathcal{J}\} = 4\mathcal{J}_{\boldsymbol{M}}{}^K\mathcal{J}_{K\boldsymbol{M}} = -4\mathcal{G}_{\boldsymbol{M}\boldsymbol{M}} = 0 \quad \text{(! because of antisymmetrization)} \tag{C.76}$$

$$\{\mathcal{J},\{\mathcal{J},\mathbf{d}\mathfrak{v}\}\} = \mathcal{J}_{\boldsymbol{M}}{}^K\mathcal{J}_{[K|}{}^L(\mathbf{d}\mathfrak{v})_{L|\boldsymbol{M}]} \neq -4\mathbf{d}\mathfrak{v} \tag{C.77}$$

As mentioned earlier, the Dorfman bracket (C.31) used in our integrability condition is just the derived bracket of the algebraic bracket. i.e. we have

$$\begin{aligned}[\mathfrak{v},\mathfrak{w}] &= [\mathbf{d}\mathfrak{v},\mathfrak{w}]^\Delta & \text{(C.78)}\\ &= [\mathbf{d}\mathfrak{v},\mathfrak{w}]^\Delta_{(1)} + \underbrace{\sum_{p\geq 2}[\mathbf{d}\mathfrak{v},\mathfrak{w}]^\Delta_{(p)}}_{=0} & \text{(C.79)}\\ &= \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} & \text{(C.80)}\end{aligned}$$

where the differential $\mathbf{d}$ has to be understood in the extended sense of (2.9), (2.32), namely as Poisson-bracket with the BRST-like generator

$$\boldsymbol{o} = \mathbf{t}^M p_M = \boldsymbol{c}^m p_m \overset{\text{locally}}{=} \mathbf{d}(x^m p_m) = -\mathbf{d}(\boldsymbol{c}^m \boldsymbol{b}_m) \tag{C.81}$$

$$p_M \equiv (p_m, 0) \tag{C.82}$$

$$\mathbf{d}\mathfrak{v} \equiv \{\boldsymbol{o},\mathfrak{v}\} = \partial_{\boldsymbol{M}} v_{\boldsymbol{M}} + \mathfrak{v}^K p_K \tag{C.83}$$

where p_m is the conjugate variable to x^m. We can now rewrite the integrability condition (C.57) as

$$\{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\{\mathcal{J},\mathfrak{w}\}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathfrak{w}\}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} \overset{!}{=} 0 \tag{C.84}$$

Remember that the Poisson bracket is a graded one, and $\mathfrak{v},\mathfrak{w}$ and $\mathbf{d}$ are odd, while $\mathcal{J}$ is even.

Let us now start with applying Jacobi to the second term of (C.84)

$$-\frac{1}{4}\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\{\mathcal{J},\mathfrak{w}\}\} = -\frac{1}{4}\{\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} - \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathfrak{w}\}\} \tag{C.85}$$

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so that we get

$$
\begin{aligned}
0 &\overset{!}{=} \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} \\
&= \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\{\{\mathbf{d}\mathcal{J},\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} - \frac{1}{4}\{\{\{\mathcal{J},\mathbf{d}\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} \\
&= \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\{\{\mathbf{d}\mathfrak{v},\mathcal{J}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\}
\end{aligned}
\tag{C.86}
$$

It would be nice to separate $\mathfrak{w}$ completely by moving it for the last term into the last bracket like in the first three terms. We thus consider only the last term for a moment and calculate it in two different ways (first using Jacobi for second and third bracket and after that using Jacobi for first and second bracket):

$$
\begin{aligned}
\frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} &\overset{1.}{=} \frac{1}{4}\{\mathcal{J},\{\{\mathbf{d}\mathfrak{v},\mathcal{J}\},\mathfrak{w}\}\} + \frac{1}{4}\{\mathcal{J},\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\mathfrak{w}\}\}\} \\
&= \frac{1}{4}\{\mathcal{J},\{\{\mathbf{d}\mathfrak{v},\mathcal{J}\},\mathfrak{w}\}\} - \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} \\
&\overset{2.}{=} \frac{1}{4}\{\{\mathcal{J},\mathbf{d}\mathfrak{v}\},\{\mathcal{J},\mathfrak{w}\}\} + \frac{1}{4}\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\{\mathcal{J},\mathfrak{w}\}\}\} \\
&= \frac{1}{4}\{\mathcal{J},\{\{\mathcal{J},\mathbf{d}\mathfrak{v}\},\mathfrak{w}\}\} + \frac{1}{4}\{\{\{\mathcal{J},\mathbf{d}\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} - \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} \\
&= -\frac{1}{4}\{\mathcal{J},\{\{\mathbf{d}\mathfrak{v},\mathcal{J}\},\mathfrak{w}\}\} + \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - 2\{\mathbf{d}\mathfrak{v},\mathfrak{w}\} + \frac{1}{4}\{\{\{\mathcal{J},\mathbf{d}\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\}
\end{aligned}
$$

Comparing both calculations yields

$$
\frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} = -\frac{1}{8}\{\{\mathcal{J},\{\mathcal{J},\mathbf{d}\mathfrak{v}\}\},\mathfrak{w}\} - \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} \tag{C.87}
$$

We can plug this back in (C.86) and leave away the outer bracket with $\mathfrak{w}$:

$$
0 \overset{!}{=} \mathbf{d}\mathfrak{v} - \frac{1}{4}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\} + \frac{1}{4}\{\{\mathbf{d}\mathfrak{v},\mathcal{J}\},\mathcal{J}\} - \frac{1}{8}\{\mathcal{J},\{\mathcal{J},\mathbf{d}\mathfrak{v}\}\} - \mathbf{d}\mathfrak{v} \tag{C.88}
$$

$$
= -\frac{1}{4}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\} + \frac{1}{8}\{\{\mathbf{d}\mathfrak{v},\mathcal{J}\},\mathcal{J}\} \tag{C.89}
$$

$$
= -\frac{1}{8}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\} + \frac{1}{8}\{\mathbf{d}\{\mathfrak{v},\mathcal{J}\},\mathcal{J}\} \tag{C.90}
$$

$$
= -\frac{1}{8}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\} + \frac{1}{8}\mathbf{d}\{\{\mathfrak{v},\mathcal{J}\},\mathcal{J}\} + \frac{1}{8}\{\{\mathfrak{v},\mathcal{J}\},\mathbf{d}\mathcal{J}\} \tag{C.91}
$$

$$
= -\frac{1}{8}\{\mathfrak{v},\{\mathbf{d}\mathcal{J},\mathcal{J}\}\} - \frac{1}{2}\mathbf{d}\mathfrak{v} \tag{C.92}
$$

$$
= \frac{1}{8}\Big(\{[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)},\mathfrak{v}\} - 4\mathbf{d}\mathfrak{v}\Big) \tag{C.93}
$$

$$
= \frac{1}{8}\{[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} - 4\boldsymbol{o},\mathfrak{v}\} \tag{C.94}
$$

where we used

$$
\mathbf{d}\mathfrak{v} = \{\boldsymbol{o},\mathfrak{v}\} \tag{C.95}
$$

The integrability condition is thus (explaining the normalization of $\mathcal{N}$ of above) as promised in (C.72)

$$
\mathcal{N} \equiv [\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} - 4\boldsymbol{o} \overset{!}{=} 0 \tag{C.96}
$$

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The derived bracket $[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)}$ indeed contains the term $4\boldsymbol{o} = 4\mathbf{t}^M p_M$ which therefore is exactly cancelled.

Precisely the same calculation can be performed by calculating with the complete algebraic bracket $[\, ,\,]^{\Delta}$ instead of the Poisson-bracket, its first order part. Similarly to above, we have

$$\mathcal{J}\mathfrak{v} \equiv \frac{1}{2}[\mathcal{J},\mathfrak{v}]^{\Delta} \tag{C.97}$$

$$\Rightarrow [\mathcal{J},[\mathcal{J},\mathfrak{v}]^{\Delta}]^{\Delta} = 4\mathcal{J}^2\mathfrak{v} = -4\mathfrak{v} \tag{C.98}$$

In combination with (C.78) this is enough to redo the same calculation and get as integrability condition (using $[\mathcal{J},\mathcal{J}] \equiv -[\mathbf{d}\mathcal{J},\mathcal{J}]^{\Delta}$)

$$\mathcal{N} \equiv [\mathcal{J},\mathcal{J}] - 4\boldsymbol{o} \stackrel{!}{=} 0 \tag{C.99}$$

which also proves that the derived bracket bracket of the big bracket (which is not necessarily geometrically well defined) coincides in this case with the complete derived bracket

$$[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} = [\mathcal{J},\mathcal{J}] \tag{C.100}$$

As discussed in (B.50) and (B.52), throwing away the $\mathbf{d}$-closed part corresponds to taking Buttin's bracket instead of the derived one. Remember that $\boldsymbol{o} = \mathbf{d}x^k p_k = -\mathbf{d}(\mathbf{d}x^k \wedge \boldsymbol{\partial}_k)$, s.th. $\mathbf{d}\boldsymbol{o} = 0$. We can thus equally write

$$\mathcal{N} = [\mathcal{J},\mathcal{J}]_B \tag{C.101}$$

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